Supplemental document accompanying submission to Optica

Title: Efficient Distribution of High-Dimensional Entanglement through 11 km Fibers

Authors: Bi-Heng Liu, Xiaomin Hu, Wen-Bo Xing, De-Yong He, Huan Cao, Yu Guo, Chao Zhang, Hao Zhang, YunFeng Huang, Chuanfeng Li, Guang-can Guo

Submitted: 1/19/2020 1:34:59 AM



Efficient Distribution of High-Dimensional Entanglement through 11 km Fiber: supplementary material

XIAO-MIN $Hu^{1,2}$, Wen-Bo Xing^{1,2}, BI-Heng Liu^{1,2,†}, De-Yong He^{1,2}, Huan Cao^{1,2}, Yu Guo^{1,2}, Chao Zhang^{1,2}, Hao Zhang^{1,2}, Yun-Feng Huang^{1,2}, Chuan-Feng Li^{1,2,‡}, and Guang-Can Guo^{1,2}

¹CAS Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei, 230026, People's Republic of China ²CAS Center For Excellence in Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, 230026, People's Republic of China

⁺Corresponding author: bhliu@ustc.edu.cn [‡]Corresponding author: cfli@ustc.edu.cn

Compiled June 2, 2020

This document provides supplementary information to "Efficient distribution of high-dimensional entanglement through 11 km fiber," https://doi.org/10.1364/OPTICA.388773.

1. BIPARTITE WITNESS FOR *D*-DIMENSIONAL ENTAN-GLEMENT

We performed quantum tomography and obtained the fidelity F after distribution. We then employed the method developed in previous work [1] to certify the 4-dimensional entanglement. In the following text, we provide a maximal overlap between the chosen high-dimensional state and states with a bounded Schmidt rank d. If the fidelity reveals a higher overlap than this bound, the justification of at least (d + 1)-dimensional entanglement is demonstrated.

The Schmidt decomposition of high dimensional state is described as $|\varphi\rangle = \sum_{i=1}^{d} \lambda_i |ii\rangle$, with the coefficients in decreasing order $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_d|$. The witness for the *d*-dimensional entanglement is constructed by comparing the two fidelities,

$$F = \operatorname{Tr}(\rho |\varphi\rangle \langle \varphi|),$$

$$F_d = \max_{|\phi_d\rangle} |\langle \phi_d |\varphi\rangle|^2,$$
(S1)

where ρ is the density matrix after distribution and $|\phi_d\rangle = \sum_{m,n=1}^d \alpha_{mn} |mn\rangle$ represents states with a bounded Schmidt rank *d*. A global search to maximize the *F*_d convinced us that *F* > *F*_d could not be satisfied by a *d*-dimensional entangled state. In other word, the generated bipartite system is entangled at least

 $(d+1)\mbox{-}dimension.$ We rewrite the maximal overlap as

$$F_{d} = \max_{|\phi_{d}\rangle} \left| \left(\langle mn | \sum_{m,n=1}^{d} \alpha_{mn}^{*} \right) \left(\sum_{i=1}^{D} \lambda_{i} | ii \rangle \right) \right|^{2}$$
$$= \max_{|\phi_{d}\rangle} \left| \left(\sum_{i,m,n=1}^{D} \langle m | i \rangle \langle n | i \rangle \alpha_{mn}^{*} \lambda_{i} \right) \right|^{2}.$$
(S2)

We next introduce two operators of the form

$$U = c_{mn} |m\rangle \langle n|,$$

$$P_d U^* = U^*,$$
(S3)

1

where P_d is a rank *d*-projector which always exists if B^* of rank *d*, as $|\phi_d\rangle$ is also of Schmidt rank *d*. Combining these equations, we have

$$F_{d} = \max_{|\phi_{d}\rangle} \left| \operatorname{Tr} \left(U^{\dagger} \sum_{i=1}^{D} \lambda_{i} |i\rangle \langle i| \right) \right|^{2}$$
$$= \max_{|\phi_{d}\rangle} \left| \operatorname{Tr} \left(P_{d} U^{\dagger} \sum_{i=1}^{D} \lambda_{i} |i\rangle \langle i| \right) \right|^{2}$$
$$= \max_{|\phi_{d}\rangle} \left| \operatorname{Tr} \left(U^{\dagger} \sum_{i=1}^{D} \lambda_{i} |i\rangle \langle i| P_{d} \right) \right|^{2}.$$
(S4)

For the inner product $\langle A, B \rangle \equiv Tr(AB^{\dagger})$ taking advantage of Cauchy-Schwarz inequality $|\langle A, B \rangle|^2 \leq \langle A, A \rangle \langle B, B \rangle$, upper

Supplementary Material

bound of F_d is found to be

$$F_{d} \leq \max_{|\phi_{d}\rangle} \operatorname{Tr}\left(BB^{\dagger}\right) \operatorname{Tr}\left(P_{d} \sum_{i=1}^{D} |\lambda_{i}|^{2} |i\rangle \langle i|P_{d}\right).$$
(S5)

Because $\operatorname{Tr}(BB^{\dagger}) = \sum_{m,n=1}^{D} c_{mn} c_{nm}^* \leq 1$ and choosing $P_d = \sum_{i=1}^{d} |i\rangle\langle i|$, we get the upper bound of F_d for *d*-dimensional entangled states with a simple formula

$$F_d \le \sum_{i=1}^d |\lambda_i|^2 \,. \tag{S6}$$

Thus we find a tight bound for witness of (d + 1)-dimensional entanglement

$$F_d = \max_{|\phi_d\rangle} |\langle \phi_d | \varphi \rangle|^2 = \sum_{i=1}^d |\lambda_i|^2.$$
(S7)

For d = 4, this bound is found to be 3/4.

2. BELL AND STEERING INEQUALITIES

We used the CGLMP Bell inequality in [2] to test the non-locality of *d*=2, 3, 4 entangled states. Here, we show more details of the scenario, in which, we focus on a particular instance of two measurements per observer, A_1 or A_2 , and B_1 or B_2 . A state ρ is shared by Alice and Bob, who perform one of two measurements with equal probability, obtaining $A_x = \{M_{a|x}\}_a$ for A, $B_x = \{M_{b|x}\}_b$ for B with x, y = 1, 2 and a, b = 0, ..., d - 1.

The correlations are joint probabilities $\{p(ab|xy)\}_{a,b} \equiv \{p(A_x = a, B_y = b)\}$. These joint probabilities mean that Alice and Bob obtain outcomes *a* and *b* separately. They can also be expressed by using Born rule as $\{p(ab|xy)\}_{a,b} = Tr[\rho(M_{a|x} \otimes M_{b|y})]$.

The Bell inequalities we used to test nonlocality of the highdimensional entangled state are:

$$I_{d} \equiv \sum_{k=0}^{\lfloor d/2 \rfloor - 1} (1 - \frac{2k}{d-1}) \{ [P(A_{1} = B_{1} + k) + P(B_{1} = A_{2} + k + 1) + P(A_{2} = B_{2} + k) + P(B_{2} = A_{1} + k)] - [P(A_{1} = B_{1} - k - 1) + P(B_{1} = A_{2} - k) + P(A_{2} = B_{2} - k - 1) + P(B_{2} = A_{1} - k - 1)] \},$$
(S8)

in which

$$P(A_x = B_y + k) \equiv \sum_{a=0}^{d-1} P(A_x = a, B_y = a + k \mod d).$$

The bases $|a\rangle_x$, $|b\rangle_y$ used by Alice and Bob is

$$|a\rangle_{x}^{d} = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{i\frac{2\pi}{d}k(a+\alpha_{x})} |k\rangle,$$

$$|b\rangle_{y}^{d} = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{i\frac{2\pi}{d}k(-b+\beta_{y})} |k\rangle,$$

(S9)

where $\alpha_1 = 0$, $\alpha_2 = 1/2$, $\beta_1 = 1/4$, and $\beta_2 = -1/4$.

To verify steering nonlocality, we introduce two sets of *d*dimensional mutually unbiased measurement bases for the two local subsystems: $|m\rangle^{A(B)} \in \{|\ell_0\rangle, \dots, |\ell_k\rangle, \dots, |\ell_{d-1}\rangle\}$ and $|M\rangle^{A(B)} \in \{|L_0\rangle, \dots, |L_k\rangle, \dots, |L_{d-1}\rangle\}$: is the superposition of $|\ell_k\rangle$: $|L_k\rangle = \sum_{j=0}^{d-1} \exp[(i2\pi/d)kj] |\ell_j\rangle / \sqrt{d}$. The steering parameter can be expressed as

$$S_{d} \equiv \sum_{i=j=0}^{d-1} P\left(\ell_{i}^{A}, \ell_{j}^{B}\right) + \sum_{i+j=0}^{d-1} P\left(L_{i}^{A}, L_{j}^{B}\right), \qquad (S10)$$

where $i + j \doteq 0$ under the second sum sign denotes equality modulo d, i.e., i + j = 0 or i + j = d, and $P\left(\ell_i^A, \ell_j^B\right)$, denotes the probability that Alice is holding particle in state $|\ell_i^A\rangle$, and in the meantime, Bob's state is $|\ell_j^B\rangle$, and the same to $P\left(L_i^A, L_j^B\right)$. For all unsteerable states, their results would be limited at $S_{\Phi}^{(\text{EPR})} \leq \alpha_{\text{bound}}$, where $\alpha_{\text{bound}} = 1 + 1/\sqrt{d}$. Therefore, if Alice could demonstrate a violation of this inequality, Bob has to admit her steerability.



Fig. S1. Measurement Setup. All HWPs are set at 22.5°. By adjusting the phase of liquid crystal (LC1-LC3), we can complete the measurement of $(|0\rangle + e^{i\varphi_1}|1\rangle + e^{i\varphi_3}(|2\rangle + e^{i\varphi_2}|3\rangle))/2$.

3. CONSTRUCTION OF MEASUREMENT BASES

Here we provide the details of the measurement setup. Let us take a four-dimensional measurement as an example. The measurement bases for the four-dimensional CGLMP inequality is as follows:

$$A_1$$
:

$$|0\rangle_{1}^{4} = (|0\rangle + |1\rangle + |2\rangle + |3\rangle)/2,$$

$$|1\rangle_{1}^{4} = (|0\rangle + i|1\rangle - |2\rangle - i|3\rangle)/2,$$

$$|2\rangle_{1}^{4} = (|0\rangle - |1\rangle + |2\rangle - |3\rangle)/2,$$

$$|3\rangle_{1}^{4} = (|0\rangle - i|1\rangle - |2\rangle + i|3\rangle)/2.$$

 A_2 :

$$\begin{split} |0\rangle_{2}^{4} &= (|0\rangle + e^{\pi i/4}|1\rangle + i|2\rangle + e^{3\pi i/4}|3\rangle)/2, \\ |1\rangle_{2}^{4} &= (|0\rangle + e^{3\pi i/4}|1\rangle - i|2\rangle + e^{\pi i/4}|3\rangle)/2, \\ |2\rangle_{2}^{4} &= (|0\rangle + e^{-3\pi i/4}|1\rangle + i|2\rangle + e^{-\pi i/4}|3\rangle)/2, \\ |3\rangle_{2}^{4} &= (|0\rangle + e^{-\pi i/4}|1\rangle - i|2\rangle + e^{-3\pi i/4}|3\rangle)/2. \end{split}$$

 B_1 :

$$\begin{split} |0\rangle_{1}^{4} &= (|0\rangle + e^{\pi i/8}|1\rangle + e^{\pi i/4}|2\rangle + e^{3\pi i/8}|3\rangle)/2, \\ |1\rangle_{1}^{4} &= (|0\rangle + e^{-3\pi i/8}|1\rangle + e^{-3\pi i/4}|2\rangle + e^{7\pi i/8}|3\rangle)/2, \\ |2\rangle_{1}^{4} &= (|0\rangle + e^{-7\pi i/8}|1\rangle + e^{\pi i/4}|2\rangle + e^{-5\pi i/8}|3\rangle)/2, \\ |3\rangle_{1}^{4} &= (|0\rangle + e^{5\pi i/8}|1\rangle + e^{-3\pi i/4}|2\rangle + e^{-\pi i/8}|3\rangle)/2. \end{split}$$

 B_2 :

$$\begin{split} |0\rangle_{2}^{4} &= (|0\rangle + e^{-\pi i/8}|1\rangle + e^{-\pi i/4}|2\rangle + e^{-3\pi i/8}|3\rangle)/2, \\ |1\rangle_{2}^{4} &= (|0\rangle + e^{-5\pi i/8}|1\rangle + e^{3\pi i/4}|2\rangle + e^{\pi i/8}|3\rangle)/2, \end{split}$$

Supplementary Material

$$\begin{aligned} |2\rangle_{2}^{4} &= (|0\rangle + e^{7\pi i/8}|1\rangle + e^{-\pi i/4}|2\rangle + e^{5\pi i/8}|3\rangle)/2, \\ |3\rangle_{2}^{4} &= (|0\rangle + e^{3\pi i/8}|1\rangle + e^{3\pi i/4}|2\rangle + e^{-7\pi i/8}|3\rangle)/2. \end{aligned}$$

In our experimental scheme, we use a hybrid of polarization and path coding. The angles of HWP1, HWP2, HWP3 are set at 22.5° (Fig. S1). The function of liquid crystal (LC) is to load a phase φ between H-polarized and V-polarized photon. LC1, LC2 and LC3 are loaded φ_1 , φ_2 and φ_3 between H-polarized and V-polarized photon, respectively. The $(|0\rangle + e^{i\varphi_1}|1\rangle + e^{i\varphi_3}(|2\rangle + e^{i\varphi_2}|3\rangle))/2$ measurement basis can be constructed using our setup (Fig. S1).

4. MULTIPLE OUTCOME MEASUREMENT BASES FOR HIGH-DIMENSIONAL QKD AND CGLMP INEQUALITY



Fig. S2. Multiple outcome measurement setups for 3dimensional and 4-dimensional QKD. (A) is the setup of 3dimensional computational basis and Fourier basis. By controlling the presence or absence of half-wave voltage loaded on LCs, two kinds of measurement bases can be switched. (B) is the setup of 4-dimensional computational basis and Fourier basis.

For high-dimensional QKD and CGLMP inequalities, multiple outcome projection measurements are generally required. In this section, we give the scheme of multiple outcome measurement bases for high-dimensional QKD and CGLMP inequalities.

Computational $(|0\rangle, |1\rangle, ..., |d-1\rangle)$ and Fourier bases $(\sum_{k=0}^{d-1} e^{i\frac{2\pi}{d}} |i\rangle/\sqrt{d})$ are needed for high-dimensional QKD. We need to set LCs at some fixed angles as shown in Fig. S2. When LC is loaded with zero voltage, LC does not affect the photon's polarization. At this time, the measurement of the computational basis is completed. When LC is loaded with half-wave voltage, the LC's effect is equivalent to an HWP, and the measurement of the Fourier basis is completed. By controlling the voltage of LCs, we can realize the fast switch of these two bases.

For CGLMP inequality, its measurement basis is a set of highdimensional MUB (Eq. S9). As shown in Fig. S3, we first construct a set of fixed Fourier bases using HWPs, QWPs, BDs,



Fig. S3. Multiple outcome measurement bases setup for 3dimensional and 4-dimensional CGLMP inequality. (A) is the setup of 3-dimensional MUB (Eq. S9). Within the dotted box is a fixed 3-dimensional multiple outcome Fourier basis. By using LCs to load different phases on different paths, different MUBs can be obtained. (B) is the setup of 4-dimensional MUB (Eq. S9).

and PBSs. After that, by adjusting the phase of each dimension through LCs, the switch of other MUBs can be completed.

5. MULTIDIMENSIONAL QUANTUM KEY DISTRIBUTION

In our experiment we use a generalized entanglement-based version of the BB84 protocol for high dimensional systems, initially proposed and analyzed in Ref. [3], and test it for different dimensions (d = 2, 3, 4). In Table S1, we report experimental correlations data for high-dimensional mutually unbiased bases required for QKD (computational and Fourier bases). The final secret key rate (per coincidence) can be obtained as [4]:

$$R^{d} = \log_2 d - 2H_d (1 - F^{QKD}), \tag{S11}$$

where $H_d(e)$ represents the d-dimensional Shannon entropy, given as a function of the fildelity F^{QKD} and the dimension d by $H_d(e) = -(1-e)\log_2(1-e) - e\log_2[e/(d-1)]$.

In Table S1, we report the maximal values of the QBER, given by the infidelity $QBER = 1 - F^{QKD}$, required to achieve a positive key rate in the cases where Eve is allowed coherent attacks $QBER_{Th}^{Coh}$. It can be observed that in general higher dimensionality corresponds to higher tolerance to noise and higher photon information efficiency.

6. PHASE STABILITY AND FIBER LOCKING SYSTEM

In our experiment, we use 11 *km* multicore fiber (MCF) to distribute path-polarization entanglement. There were two challenges, one is the polarization preservation through long-distance fibers, the other is the phase stability between different cores. Firstly, we lay all the fibers including fan-in and fan-out on the optical table. We place an HWP in each export of the fan-in/fan-out to maintain H- and V-polarization, and a tilt quarterwave plate to compensate for the differential birefringence of

Table S1. Security analysis for a device-dependent QKD (BB84-type) in different dimensions. Reported are the experimentally measured fidelity F^{QKD} , the associated experimental $QBER_{Exp}$, the theoretical maximal QBER bounds $QBER_{Th}^{Coh}$ for coherent attacks, experimental secure key rate R_{Exp} and theoretical secure key rate bound R_{Bound} from dimension 2 to 4.

d	$F^{QKD}(\%)$	$QBER_{Exp}(\%)$	$QBER_{Th}^{Coh}(\%)$	$R_{Exp}(bpc)$	$R_{Bound}(bpc)$
2	98.30 ± 0.04	1.59 ± 0.04	≤ 11.00	0.756 ± 0.03	1
3	96.40 ± 0.06	2.84 ± 0.06	≤ 16.00	1.062 ± 0.04	1.584
4	94.50 ± 0.07	5.53 ± 0.07	≤ 18.93	1.268 ± 0.04	2



Fig. S4. Polarization stabilities of single cores. Here, the average visibility in an hour is 0.985 ± 0.001 and 0.969 ± 0.001 for core1 and core2, respectively.

the two cores. In the laboratory environment (the temperature fluctuation in the laboratory is $\pm 0.5^{\circ}C$), the polarization states are maintained well as shown in Fig. S4. The average visibilities in an hour are 0.985 \pm 0.001 and 0.969 \pm 0.001 for core1 and core2. These results are worse than those using single-mode fibers in a length of a few meters due to the polarization mode dispersion and group velocity dispersion. For an outdoor environment, an active feedback system is feasible to compensate for the polarization mode dispersion [5]. Then, we use a fiber locking system (FLS) to lock the relative phase between core1 and core2 used in our experiment. Fig. S5 shows the phase drift between the two cores without active feedback. Different from Ref [6], where the phase drift is very slow in 300 *m* MCFs (the time scale of 2π phase change is in an order of hours without active feedback), the phase drift in our experiment is terrible without feedback. The difference is caused by the coupling methods. In our experiment, we use packaged fan-in and fan-out where the spatially separated fiber pigtails are approximate 10 m in length, and the phase drift between different paths is increased dramatically. Combined with 11 km MCF, the time scale of 2π phase change between different paths is in an order of minutes without active feedback as shown in Fig. S5. As shown in Fig. S6, the system is stable after the active feedback system (FLS given in Fig. 1, the frequency of the feedback system is 5 Hz) is turned on. To test the feedback system, we measured the visibility between different cores (core1-core2 and core1-core3), and the average visibility of the two cases in one hour is $V = 0.915 \pm 0.001$, and 0.907 ± 0.001 . These results prove that it is feasible to use more cores to distribute high-dimensional path entanglement. The

brightness of reference light used in the locking system is 10^6 photons/s.



Fig. S5. Phase drift between core1 and core2 without active feedback. Due to the influence of fan-in and fan-out pigtails, the phase drift is faster than that of the multicore fiber itself [6]. Here, $P = N_{Detection}/N_{Total}$ refers to the proportion of photons detected in one arm of the interferometer. To reduce the fluctuation, we use a total photon rate of 10^6 photons/s.

REFERENCES

- R. Fickler, R. Lapkiewicz, M. Huber, M. P. Lavery, M. J. Padgett, and A. Zeilinger, Interface between path and orbital angular momentum entanglement for high-dimensional photonic quantum information. *Nat. Commun.* 5, 4502 (2014).
- D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Bell inequalities for arbitrarily high-dimensional systems. *Phys. Rev. Lett.* 88, 040404 (2002).
- N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, Security of quantum key distribution using d-level systems. *Phys. Rev. Lett.* 88, 127902 (2002).
- F. X. Wang, W. Chen, Z. Q. Yin, S. Wang, G. C. Guo, and Z. F. Han, Characterizing high-quality high-dimensional quantum key distribution by state mapping between different degrees of freedom. *Phys. Rev. A* 11, 024070 (2019).
- C. Z. Peng, J. Zhang, D. Yang, W. B. Gao, H. X. Ma, H. Yin, H. P. Zeng, T. Yang, X. B. Wang, and J. W. Pan, Experimental long-distance decoy-state quantum key distribution based on polarization encoding. *Phys. Rev. Lett.* 98, 010505 (2007).



Fig. S6. Interference visibility between different cores with active feedback. The brown and green lines represent the visibility of core1-core2, core1-core3, respectively. Here core1, core2 and core3 are selected randomly (we have no details about the cores, here we number the cores randomly).

 G. Canas *et al.* High-dimensional decoy-state quantum key distribution over multicore telecommunication fibers. *Phys. Rev. A* 96, 022317 (2017).