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Fundamental limits of quantum illumination: supplementary material

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This document provides supplementary information to "Fundamental limits of quantum illumination," https://doi.org/10.1364/OPTICA.391335. Supporting calculations for the results on detection of Rayleigh fading targets and on target reflectance estimation are presented.

1. TARGETS EXHIBITING FLAT RAYLEIGH FADING: ERROR PROBABILITY LOWER BOUND FOR QI

We asserted in the main text that, for targets exhibiting flat Rayleigh fading, the density operators of the joint return-idler system when the target is absent and present are given by

$$\rho_0 = \left[\mathrm{id}_I \otimes \left(\mathcal{L}_{0,N_B}^{\otimes M}\right)\right](\Psi), \tag{S1}$$

$$\rho_1 = (1/2\pi)\int_0^1 \mathrm{d}\eta\, P(\eta) \int_0^{2\pi} \mathrm{d}\phi \left[\mathrm{id}_I \otimes \left(\mathcal{U}_\phi \circ \mathcal{L}_{\eta, N_B^{(\eta)}}\right)^{\otimes M}\right](\Psi) \tag{S2}$$

respectively. It is usual in the classical radar literature to assume that $\sqrt{\eta}$ has a Rayleigh distribution – see, e.g., Sec. 4.4.2 of [1]. Then η itself has the exponential probability density $\widetilde{P}(\eta) = (1/\overline{\eta})\exp(-\eta/\overline{\eta})$ supported on $\eta \geqslant 0$. Strictly speaking, the probability that $\eta > 1$ should be zero since the target is a passive reflector. However, the above model is an excellent approximation for a diffuse reflector as long as $\overline{\eta} \ll 1$, which is usually the case in practice.

Quantum mechanically, however, Eq. (1) of the main text does not represent a physically possible transformation if $\eta > 1$. To deal with this issue, we replace $\widetilde{P}(\eta)$ with the truncated exponential density

$$P(\eta) = \begin{cases} \exp(-\eta/\overline{\eta}) \,/\, \left[\overline{\eta}\left(1-e^{-1/\overline{\eta}}\right)\right] & \text{if } \eta \in [0,1] \\ 0 & \text{if } \eta \geqslant 1. \end{cases} \tag{S3}$$

Again, if $\overline{\eta} \ll 1$, the discrepancy between Eq. (S3) and $\widetilde{P}(\eta)$ is negligible. It is the probability density of Eq. (S3) that appears in Eq. (S2) and Eq. (15) of the main text. Finally, note that setting $N_B^{(\eta)} = N_B/(1-\eta)$ in Eq. (S2) enforces the no-passive-signature assumption in this fading scenario. While this implies that $N_B^{(\eta)}$ can vary greatly in the vicinity of $\eta \approx 1$, such large deviations of the background noise in the model have very low probability if $\overline{\eta} \ll 1$.

We can now proceed to develop our error probability lower bound. First, we observe that the squared fidelity $F^2(\rho,\sigma)$, like $F(\rho,\sigma)$ itself [2], is concave in each of its arguments [3], so that we can write

$$\begin{aligned} F^2(\rho_0,\rho_1) \geqslant (1/2\pi)\int_0^1 \mathrm{d}\eta\, P(\eta)\int_0^{2\pi}\mathrm{d}\phi \\ \times F^2\left\{\rho_0, \left[\mathrm{id}_I \otimes \left(\mathcal{U}_\phi \circ \mathcal{L}_{\eta,N_B^{(\eta)}}\right)^{\otimes M}\right](\Psi)\right\}. \end{aligned} \tag{S4}$$

Noting that the fidelity appearing in the integrand is ϕ-independent, we can apply the inequalities of Eqs. (10)-(11) of the main text to it and use the bound $P_e[\sigma_0,\sigma_1] \geqslant \pi_0\pi_1 F^2(\sigma_0,\sigma_1)$ to get the lower bound

$$P_e^{\Psi;\text{fading}} \geqslant \pi_0\pi_1 \int_0^1 \mathrm{d}\eta\, P(\eta)\left[\sum_{n=0}^{\infty} p_n\left(1-\frac{\eta}{N_B+1}\right)^{n/2}\right]^2 \tag{S5}$$

on the average error probability of detecting a fading target. For any given transmitter Ψ with corresponding $\{p_n\}$, the right-hand side can be evaluated analytically in some cases, and numerically otherwise.

We can further derive an analytical transmitter-independent bound as follows. Applying Jensen's inequality to the quantity in brackets in Eq. (S5) gives

$$P_e^{\Psi;\text{fading}} \geqslant \pi_0\pi_1 \int_0^1 \mathrm{d}\eta\, P(\eta) \left(1 - \frac{\eta}{N_B+1}\right)^{\mathcal{N}_S}. \tag{S6}$$

For $N_B > 0$ and $0 \leqslant \eta \leqslant 1$, we have $1 - \eta/(N_B+1) \geqslant \exp(-\gamma\eta)$, where $\gamma = \ln(1 + 1/N_B)$ is chosen such that the graph of $\exp(-\gamma\eta)$ intersects that of $1 - \eta/(N_B+1)$ at $\eta = 0$ and $\eta = 1$. Substituting this lower bound into Eq. (S6) and evaluating the integral gives

$$P_e^{\text{QI;fading}} \geqslant \pi_0\pi_1 \frac{1-\exp\left(-\gamma\mathcal{N}_S - 1/\overline{\eta}\right)}{\left[1-\exp\left(-1/\overline{\eta}\right)\right]\left(1+\overline{\eta}\gamma\mathcal{N}_S\right)}, \tag{S7}$$

$$\geqslant \frac{\pi_0\pi_1}{1+\overline{\eta}\gamma\mathcal{N}_S}, \tag{S8}$$

which is Eq. (16) of the main text.

2. ESTIMATION OF TARGET REFLECTANCE

In this section, we provide derivations of the results pertaining to estimating the reflectance $\eta \ll 1$ of a weakly reflecting specular target. As described in the main text, for any transmitter Ψ, the density operator ρ_η of the returned signal and idler modes conditioned on the target reflectance having the value η is given by

$$\rho_\eta = \left[\mathrm{id}_I \otimes \left(\mathcal{U}_\phi^{\otimes M} \circ \mathcal{L}^{\otimes M}_{\eta, N_B^{(\eta)}}\right)\right](\Psi), \tag{S9}$$

$$= \left[\mathrm{id}_I \otimes \left(\mathcal{U}_\phi^{\otimes M} \circ \mathcal{A}^{\otimes M}_{N_B+1} \circ \mathcal{L}^{\otimes M}_{\eta/(N_B+1)}\right)\right](\Psi), \tag{S10}$$

where we have used the decomposition of Eq. (8) of the main text. Now note that the quantum channel $\mathcal{U}_\phi^{\otimes M} \circ \mathcal{A}^{\otimes M}_{N_B+1}$ that is applied 'downstream' to the S system is η-independent, and can be realized by coupling an ancilla mode A in a fixed state to the S system and evolving the joint system under a fixed unitary (this is the so-called Stinespring dilation of a quantum channel [2]). The monotonicity property of the QFI under partial trace [4] then implies that the QFI on η achieved by making a measurement on the joint ISA system is at least as much as that on the IS system alone. On the other hand, the invariance of QFI under a known η-independent unitary transformation implies that the former value equals the QFI on η of the state family

$$\sigma_\eta = \left[\mathrm{id}_I \otimes \mathcal{L}^{\otimes M}_{\eta/(N_B+1)}\right](\Psi). \tag{S11}$$

We have thus reduced the problem to maximizing the QFI on η for the outputs $\{\sigma_\eta\}$ of *pure-loss* channels under an energy constraint on the S modes. This problem was solved in [5] (cf. Eq. (14) therein), and transforming variables in that result gives the upper bound

$$\mathcal{K}_\eta^{\text{QI}} \leqslant \frac{\mathcal{N}_S}{\eta\left(N_B + 1 - \eta\right)} \tag{S12}$$

for the QFI of any transmitter Ψ for any value of the excess noise N_B, which reproduces Eq. (19) of the main text.

Consider a single-mode coherent-state transmitter $|\psi\rangle_S = |\sqrt{\mathcal{N}_S}\rangle_S$ of energy $\mathcal{N}_S$. In order to evaluate the QFI on η, we first calculate the fidelity between the states ρ_η^{CS} and $\rho_{\eta'}^{\text{CS}}$ of Eq. (S9) for any two values η and η'. Using known results on the fidelity between Gaussian states (see e.g., Eq. (3.7) of [6]), we have

$$F\left(\rho_\eta^{\text{CS}}, \rho_{\eta'}^{\text{CS}}\right) = \exp\left[-\frac{\left(\sqrt{\eta'} - \sqrt{\eta}\right)^2 \mathcal{N}_S}{4N_B + 2}\right] \tag{S13}$$

The QFI then follows as

$$\mathcal{K}_\eta^{\text{CS}} = -4 \left.\frac{\partial^2 F\left(\rho_\eta^{\text{CS}}, \rho_{\eta'}^{\text{CS}}\right)}{\partial \eta'^2}\right|_{\eta'=\eta}$$

$$= \frac{\mathcal{N}_S}{\eta(2N_B+1)}. \tag{S14}$$

The additivity of the QFI for product states [4] and the linearity of the coherent-state QFI (S14) in the energy imply that (S14) is also the QFI of a multimode coherent state of total energy $\mathcal{N}_S$. Finally, any classical-state transmitter can be written as a proper P-representation [7], i.e., in the form

$$\rho = \int_{\mathbb{C}^M} \mathrm{d}^{2M}\boldsymbol{\alpha}_I \int_{\mathbb{C}^M} \mathrm{d}^{2M}\boldsymbol{\alpha}_S\, P(\boldsymbol{\alpha}_I, \boldsymbol{\alpha}_S)\, |\boldsymbol{\alpha}_I\rangle\langle\boldsymbol{\alpha}_I|_I \otimes |\boldsymbol{\alpha}_S\rangle\langle\boldsymbol{\alpha}_S|_S, \tag{S15}$$

where $\boldsymbol{\alpha}_S = \left(\alpha_S^{(1)}, \ldots, \alpha_S^{(M)}\right) \in \mathbb{C}^M$ indexes M-mode coherent states $|\boldsymbol{\alpha}_S\rangle_S$ of S, $\boldsymbol{\alpha}_I = \left(\alpha_I^{(1)}, \ldots, \alpha_I^{(M)}\right) \in \mathbb{C}^M$ indexes M-mode coherent states $|\boldsymbol{\alpha}_I\rangle_S$ of I, and $P\left(\boldsymbol{\alpha}_I, \boldsymbol{\alpha}_S\right) \geqslant 0$ is a probability distribution. An average signal energy constraint of $\mathcal{N}_S$ implies that $P\left(\boldsymbol{\alpha}_I, \boldsymbol{\alpha}_S\right)$ should satisfy

$$\int_{\mathbb{C}^M} \mathrm{d}^{2M}\boldsymbol{\alpha}_I \int_{\mathbb{C}^M} \mathrm{d}^{2M}\boldsymbol{\alpha}_S\, P(\boldsymbol{\alpha}_I, \boldsymbol{\alpha}_S) \left(\sum_{m=0}^{M} \left|\alpha_S^{(m)}\right|^2\right) = \mathcal{N}_S. \tag{S16}$$

The convexity of the QFI [8], its invariance under adjoining an idler system in an η-independent state, and the linearity of the QFI (S14) in the energy then imply that the QFI of any classical probe Eq. (S15) obeying the constraint Eq. (S16) satisfies

$$\mathcal{K}_\eta^{\text{cl}} \leqslant \frac{\mathcal{N}_S}{\eta(2N_B+1)}, \tag{S17}$$

which is Eq. (20) of the main text.

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