Supplemental Document



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# Broadband thermomechanically-limited sensing with an optomechanical accelerometer: supplemental document

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# S1. Harmonic Oscillator Model

A major benefit of the accelerometer described in the article is that its dynamic response closely follows that of a onedimensional viscously-damped harmonic oscillator, making it possible to convert from measured proof mass displacement to an equivalent acceleration using a low-order model. In this section, we describe the harmonic oscillator model and the conversion between displacement and acceleration. Much of the analysis in this section and the next follows directly from the work of Gabrielson [S1] but is specifically focused towards the optomechanical accelerometer.

The harmonic oscillator model is described in Fig. S1, where a mass-spring-damper system is driven by a base excitation,  $x_e$ . A stochastic force,  $F_L$ , is also applied to the harmonic oscillator, which results in Brownian motion, generating thermomechanical displacement noise. The oscillator can be described by the following Langevin equation

$$m\ddot{x} + c\left(\dot{x} - \dot{x}_e\right) + k\left(x - x_e\right) = F_L \tag{S1}$$

where *m* is the mass, *k* is the spring stiffness, *c* is the damping coefficient, and *x* is the displacement of the mass. Defining the change in optical cavity length,  $x_c$ , as  $x_c = x - x_e$  and the base acceleration,  $a_e$ , as  $a_e = \ddot{x}_e$  results in the model of interest:

$$\ddot{x}_{c} + \frac{\omega_{0}}{Q} \dot{x}_{c} + \omega_{0}^{2} x_{c} = -a_{e} + \frac{F_{L}}{m}$$
(S2)

where  $\omega_0 = \sqrt{k/m}$ ,  $\omega_0 = 2\pi f_0$ ,  $f_0$  is the resonance frequency in the absence of damping,  $Q = m\omega_0/c$ , and Q is the quality factor.

The relationship between cavity displacement,  $x_c$ , and base acceleration,  $a_e$ , as a function of frequency,  $\omega$ , can be determined from eq. (S2) by neglecting the Langevin force,  $F_L$ .

$$x_{c}(\omega) = \frac{-1}{\omega_{0}^{2} - \omega^{2} - i\frac{\omega_{0}\omega}{O}}a_{e}(\omega) = G(i\omega)a_{e}(\omega) \quad (S3)$$

The amplitude of  $a_e$  can then be written as

$$\left|a_{e}\left(\omega\right)\right| = \left|G(i\omega)\right|^{-1} \left|x_{c}\left(\omega\right)\right|, \qquad (S4)$$



**Fig. S1** Harmonic oscillator model described by a mass-springdamper system. m: mass, k: spring stiffness, c: damping coefficient, x: proof mass displacement,  $x_e$ : base displacement,  $F_L$ : Langevin force.

which has been used to calculate the acceleration data in Figs. 3d and 4d in the article from displacement measurements. Implementing eq. (S4) requires measurement of  $\omega_0$  and Q. Here, this was done by applying a least-squares fit of  $|G(i\omega)|$  to the data in Figs. 3b and 3c in the article.

#### S2. Thermomechanical and Optical Shot Noise

The stochastic force in the Langevin equation, eq. (S1), is defined as  $F_L = \sqrt{4k_BTc} \Gamma(t)$ , where  $k_B$  is Boltzmann's constant, *T* is temperature, and  $\Gamma(t)$  is a Gaussian white noise process with a standard deviation of 1 [S1]. Returning to eq. (S2), ignoring  $a_e$ , and taking the power spectral density of  $x_c$ , defined as  $S_{xx}$ , results in

$$S_{xx}(\omega) = \left|G(i\omega)\right|^2 \frac{4k_B T \omega_0}{mQ} \tag{S5}$$

The thermomechanical noise in terms of displacement is then defined as  $x_{th} = S_{xx} (\omega)^{1/2}$ , or

$$x_{th}(\omega) = |G(i\omega)| \sqrt{\frac{4k_B T \omega_o}{mQ}}$$
(S6)

Recalling the conversion from displacement to acceleration, eq. (S4), the equivalent acceleration due to thermomechanical noise is then

$$a_{th} = \sqrt{\frac{4k_B T \omega_o}{mQ}}$$
(S7)

Interestingly,  $a_{th}$  is only a function of the resonator parameters  $(\omega_0, m, \text{ and } Q)$  and temperature, and not a function of frequency, meaning that the thermomechanical noise floor in terms of acceleration is flat.

In addition to thermomechanical noise, optical shot noise is the other fundamentally limiting noise source. The power spectral density of the optical shot noise is  $S_{PP} = 2hvP_a/\eta$ , where *h* is Planck's constant, *v* is the optical frequency of the laser,  $P_a$  is the average power reaching the photodetector, and  $\eta$ is the quantum efficiency of the photodetector. This can be converted to shot noise in terms of displacement using

$$x_{s} = g_{x/V} g_{V/i} R S_{PP}^{1/2} = g_{x/V} g_{V/i} R \sqrt{2hv P_{a}/\eta}$$
(S8)

The gain  $g_{x/V}$  converts photodetector voltage to displacement and is discussed in Section S4, while  $g_{V/i}$  and R are the transimpedance gain and responsivity of the photodetector. Recalling eq. (S4), the shot noise in terms of acceleration is

$$a_{s} = g_{x/V} g_{V/i} R \sqrt{2hv P_{a}/\eta} \left| G(i\omega) \right|^{-1}$$
(S9)

Since the thermomechanical noise and shot noise are uncorrelated, they can be summed in quadrature to get the total noise equivalent displacement,  $x_{NE}$ , and acceleration,  $a_{NE}$ .

Unlike the thermomechanical displacement noise,  $x_{th}$ , the optical shot noise does not represent real resonator motion but rather, it is detection noise that is analytically referred to either displacement or acceleration. As a result, the best-case scenario for a resonator with fixed parameters ( $\omega_0, Q, m, T$ ) is for the optical shot noise to be lower than the thermomechanical noise. In this situation, the optical readout will measure the motion of the resonator with minimal contribution from shot noise. This is shown in Fig. S2, where the calculated noise floor is presented for a resonator with parameters similar to those described in the experiments in the article. Three different levels of shot noise are shown, where two are above the thermomechanical noise (dark blue, light blue) and one is below (red). When the shot noise is below the thermomechanical noise, the resonance shape is observed over the entire frequency range, which provide better estimates of  $\omega_0$  and Q when fitting displacement noise spectra to the harmonic oscillator model.

After converting the displacement to acceleration, as shown in Fig. S2b, the importance of reducing the shot noise is readily apparent. The noise equivalent acceleration is nearly flat over the frequency range when the shot noise is below the thermomechanical noise. Achieving a flat noise floor in acceleration is critical for a broadband accelerometer because it enables the measurement of signals with widely varying frequencies at the same precision level. For example, if the



Fig. S2 Noise equivalent displacement and acceleration for varying optical shot noise level. (a) Noise equivalent displacement combining thermomechanical noise and optical shot noise at three different shot noise levels.  $\omega_0 = 2\pi$  (9.8 kHz), Q = 70, m = 11 mg, T = 293 K. (b) Noise equivalent acceleration based on the displacement noise in (a).

acceleration is a square wave, all of the harmonics within the bandwidth of the sensor will be measured with the same precision when the noise floor is flat, which means that the signal can be accurately reconstructed from the data. If the noise floor is frequency dependent, this reconstruction would be less accurate since the signal-to-noise ratio will vary across the frequency range.

#### S3. Design of the Mechanical Resonator

The mechanical resonator has a large square single-crystal silicon proof mass (thickness:  $525 \,\mu$ m, width:  $3.02 \,\text{mm}$  (Device A) or  $4.02 \,\text{mm}$  (Device B)) that is supported by an array of  $1.5 \,\mu$ m thick silicon nitride beams, as shown in Fig. 1 of the article. These beams are located around the entire perimeter of the proof mass and on both sides of the chip, where the beam length is selected to achieve the desired stiffness. This design increases the resonance frequencies for rotational modes of the proof mass (i.e., rocking modes) so that there is a large separation in frequency between the first translational mode (i.e., piston mode) and the other vibrational modes.

Structural finite element analysis (FEA) was performed for the two designs (Devices A and B) to assess the effectiveness



**Fig. S3 Mode shapes for the mechanical resonator.** (a) First piston mode, and (b) first rocking mode. Red indicates maximum displacement and dark blue represents no displacement.

of mode separation due to the flexural constraints. Figure S3 shows representative mode shapes for the first piston mode and first rocking mode. The piston mode is the mode of interest for detecting accelerations perpendicular to the chip surface. This mode exhibits pure translation of the proof mass along the optical axis, such that proof mass displacement causes a length change of the optical cavity. It was found that the resonance frequency of the first rocking mode is higher than the piston mode by a factor of 11.6 for Device A and 7.8 for Device B. This mode separation is sufficient to ensure that the rocking mode does not appear within the measurement bandwidth used for Fig. 3 in the article. The closest mechanical mode detected in experiments is above 60 kHz, or a factor of 6 higher than the piston mode, as shown in Fig. S4b.

# S4. Converting from Photodetector Voltage to Displacement

Displacement of the proof mass results in a change in cavity length, which is measured by the cavity readout. With the probing laser locked to the side of a TEM<sub>00</sub> optical resonance, the cavity length change,  $\Delta L$ , is transduced by measuring the change in the center wavelength of the optical resonance,  $\Delta \lambda$ , using:

$$\Delta L = \frac{L}{\lambda} \Delta \lambda \tag{S10}$$

where L is the nominal cavity length and  $\lambda$  is the nominal laser wavelength at the lock point. The change in the center wavelength,  $\Delta\lambda$ , is related to the reflected laser intensity from the cavity that is measured with a photodetector, resulting in a voltage change,  $\Delta V$ . The relationship between voltage and wavelength is defined by the slope of the optical resonance at the locking point,  $dV/d\lambda$ , as shown in the inset of Fig. S4a. The laser was locked to the point of greatest slope for the highest transduction sensitivity. In this way, the displacement of the proof mass is found using:

$$\Delta L = \frac{L}{\lambda} \Delta V / \left(\frac{dV}{d\lambda}\right) = g_{x/V} \Delta V$$
 (S11)

The parameters  $(L, \lambda, dV/d\lambda)$  are directly found from a spectral measurement of the cavity over a full free spectral range (FSR) and the voltage change,  $\Delta V$ , is measured with an electronic spectrum analyzer (ESA).



**Fig. S4 Cavity readout with the external cavity diode laser.** (a) Schematic of the cavity readout for the accelerometer using both the external cavity diode laser (ECDL) and fiber laser (FL). EOM: electrooptical modulator, SW: switch, OSA: optical spectrum analyzer; CIR: circulator, BPD: balanced photodetector, PD: photodetector, VOA: variable optical attenuator, ESA: electronic spectrum analyzer, LPF: low-pass filter, VCO: voltage-controlled oscillator. (b) Displacement noise spectra for the accelerometer when using the ECDL and FL.

#### S5. Readout Using the External Cavity Diode Laser

Two different lasers were used for cavity readout: a continuously tunable external cavity diode laser (ECDL) and a tunable fiber laser (FL) that is phase modulated with an electrooptic modulator (EOM). The ECDL has a wide wavelength tuning range and precise piezo-based wavelength control, allowing for cavity characterization and FSR measurements, as shown in Fig. 2 of the article. In comparison, the FL has a slow tuning rate and a much narrower tuning range. Furthermore, the internal feedback locking module of the ECDL enables direct and convenient cavity displacement readout. However, the ECDL has more internal frequency noise than the FL, which appears as noise equivalent displacement. Therefore, the FL was used for the displacement noise floor measurements in Fig. 3 of the article since it has a cleaner frequency spectrum. Details on the readout method using the FL are described in the article. Here, we provide additional information on the readout with the ECDL.

As shown in Fig. S4a, the main differences between using the ECDL and FL are the wavelength tuning method and the feedback servo loop. Wavelength tuning with feedback is achieved in the ECDL with a piezoelectric actuator in the external cavity. Therefore, unlike the FL, an EOM is not needed for locking. Regarding the implementation of the servo, the ECDL has an internal digital proportional-integral-derivative (PID) feedback controller while the FL servo uses an external analog PID controller.

A comparison of the displacement noise spectra from the accelerometer is shown in Fig. S4b for both readout lasers. No mechanical resonances other than the fundamental near 10 kHz are observed in the accelerometer up to 60 kHz. In general, the responses from the two lasers are very similar. However, the ECDL exhibits several resonances near 1.3 kHz that were determined to be mechanical resonances within the external cavity of the laser. The measurements in Fig. 4 of the article were performed with the ECDL since the resulting displacements are well above the noise floor and the ECDL provides wider tuning range and simpler operation.

#### S6. Resonator Mass

The value of the proof mass in the mechanical resonator was calculated using the designed geometry and approximate densities for single-crystal silicon and the optical coatings, resulting in 11.07(53) mg for Device A and 19.59(94) mg for Device B. The main source of uncertainty in the mass is the variation in the silicon wafer thickness ( $\pm 25 \,\mu$ m) which gives a relative uncertainty of approximately 5 % for the calculated mass. This only limits the a priori estimate of the mass, not the uncertainty of the acceleration measurement, which relies on in situ measurement of  $\omega_0$  and Q.

A similar proof mass from the same fabrication process was measured for Devices A and B after being removed from the chip. The masses were calibrated by the NIST Mass and Force Group and found to be 11.13 mg for Device A and 19.88 mg for Device B, which deviate from the calculated value by 0.5% and 1.5%, respectively. Any microbeams adhering to the proof mass after removal would increase the mass by less than 20  $\mu$ g, and the uncertainty of the calibrated values [S2] is also negligible relative to the uncertainty of the calculated values.

# S7. Uncertainties in Parameters Estimated from Fits

Fitting thermomechanical noise spectra allows  $\omega_0$ , Q, and m to be measured, given the temperature. These values can vary over time due to changes in laboratory conditions, such as temperature, aging from sources including curing of packaging adhesive or accumulated stress from cycling between air and vacuum. To estimate the associated uncertainties, we use the standard deviation of multiple measurements on a device over a period of approximately eleven months. The uncertainty reported by the fitting routines is not included in the stated uncertainty as it is small compared to the variation over a year,

even when accounting for variation in fitting procedures. This represents a conservative estimate for the measurements reported here. The uncertainty can be substantially reduced, for example by measuring  $\omega_0$  and Q immediately before and after acceleration measurement, but best practice for accurate acceleration metrology with the devices is outside the scope of this work and will be reported elsewhere. For Device A the relative uncertainties for  $\omega_0$ , Q, and m are approximately 0.2 %, 2 %, and 8 %, respectively. Only the uncertainties in  $\omega_0$  and Q directly contribute to the uncertainty in acceleration measurement.

#### **S8. Homodyne Interferometer**

The homodyne Michelson interferometer used to test the accelerometer on a shaker table is shown in Fig. 4a from the article. A 632.8 nm stabilized HeNe laser is split into the measurement and reference arms of the interferometer using a non-polarizing 50/50 beam splitter. The light in the reference arm is reflected off of a piezoelectric-actuated mirror and light in the measurement arm is reflected off of a 5 mm square gold mirror mounted on the optomechanical accelerometer package. The reflected light from both arms interferes on a photodetector. The interferometer is locked to the quadrature point (i.e., point of highest fringe slope) using the piezoelectric mirror in the reference arm and a servo controller with a bandwidth below 100 Hz. Shaker vibrations above the servo bandwidth are measured with the interferometer and are converted to displacement using the measured fringe amplitude and laser wavelength, resulting in a noise floor of approximately 60 fm/ $\sqrt{\text{Hz}}$  above 1 kHz. The optomechanics for the interferometer sit on the same optical table as the shaker table, making them susceptible to vibrations driven by the shaker, as seen in the data in Fig. 4 from the article.

#### **S9.** Linearity of the Shaker Table

The comparison between the accelerometer and laser interferometer shown in Fig. 4 of the article required that the excitation amplitude of the shaker be different when using the two measurement methods. This was due to the higher sensitivity of the accelerometer relative to the interferometer by a factor of approximately 600. As a result, higher excitation amplitudes were required for detection with the interferometer. These high excitation amplitudes could not be used while reading out the microcavity in the accelerometer because the side lock could not be maintained. The end result was that measurements with the interferometer were performed with excitation amplitudes that were as much as 50 times greater than with the accelerometer readout. Due to this, the reported displacement and acceleration data are normalized by the shaker drive voltage.

This approach to the comparison is acceptable as long as the piezoelectric shaker table has a linear response for increasing excitation voltage. The linearity of the shaker table was characterized over a range of excitation voltages and frequencies, as shown in Fig. S5. The displacement of the



**Fig. S5. Linearity of the shaker table.** (a) Shaker table displacement as a function of excitation voltage at a drive frequency of 2 kHz. (b) Residuals from a linear fit to the data in (a). The residuals are an absolute value of the difference between the data and fit, expressed as a percentage of the fit value. Blue lines represent the mean (dash) and standard deviation (dash-dot) over the range of excitation voltages. (c) Mean and standard deviation residuals of the linear fit as a function of drive frequency. Blue line represents the mean over all frequencies.

shaker table for increasing excitation voltage at a single frequency (2 kHz) was found to be highly linear (Fig. S5a). The residuals for a linear fit to the data in Fig. S5a show a deviation from linearity of no more than 3 % and this deviation is much lower at higher excitation voltages due to the improved signal-to-noise ratio (Fig. S5b). Additional linearity measurements were performed between 2 kHz and 7 kHz and the mean and standard deviation of the linear fit residuals were calculated (Fig. S5c). The shaker is linear within 3 % across the entire frequency range with the exception of an outlier at 6 kHz and the mean residual is 1.1 %. This level of linearity is more than adequate for the comparison between the accelerometer and interferometer, which is discussed further in the next section.

# S10. Accelerometer and Interferometer Comparison

The data in Fig. 4c of the article was analyzed to compare the results from the accelerometer and interferometer when operating on the shaker table. The deviation of the accelerometer from the interferometer was calculated as a percentage, as indicated by the blue dots in Fig. S6. A moving average filter was applied to the data from the interferometer because noise in the data was found to be a major contributor to the deviation between the two measurements. This resulted in the black line in Fig. S6, showing a significant improvement in the comparison. The deviation for the filtered data is 5.4 %  $\pm$  15.9 % (average  $\pm$  standard deviation) over the entire drive frequency range (1 kHz to 20 kHz). When looking at a narrower



Fig. S6. Comparison of the accelerometer and interferometer results on the shaker table. Blue dots: deviation of the accelerometer results from the interferometer results. Black line: Same data set as blue dots but filtered using a moving average.

frequency range from 4.5 kHz to 11 kHz, the deviation is -0.1 %  $\pm$  9.7 %. This deviation between accelerometer and interferometer is due to a number of factors but appears to be dominated by: 1) coupling between the shaker table and optomechanics in the interferometer, 2) dynamics of the stainless-steel package, and 3) the mounting interface. Each of these will be explored in future work.

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