Supplemental Document

applied optics

Optical vortices and orbital angular momentum in strongly coupled optical fibers: supplement

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Supplement 1

Consider first a general form of matrix elements e_{13}, e_{14}, V_{13} and V_{14} :

$$I = \iint_D e^{-il(\bar{\varphi} \pm \tilde{\varphi})} F_l(\bar{r}) F_l(\tilde{r}) dS, \qquad \textbf{(S1)}$$

where *D* is some integration area and for designation see Fig. 1. Then

$$I^* = \iint_D e^{il(\bar{\varphi} \pm \tilde{\varphi})} F_l(\bar{r}) F_l(\tilde{r}) dS. \qquad \textbf{(S2)}$$

From Fig. 1 it is evident that

$$\begin{aligned} &\bar{r}^2 = y^2 + (x + L/2)^2, \tilde{r}^2 = y^2 + (x - L/2)^2, \\ &\tan \bar{\varphi} = \frac{y}{x + L/2}, \tan \tilde{\varphi} = \frac{y}{x - L/2}. \end{aligned} \qquad \textbf{(S3)}$$

From (S3) one can establish the following symmetry properties:

$$\begin{aligned} &\bar{r}(-y) = \bar{r}(y),\ \tilde{r}(-y) = \tilde{r}(y), \\ &\bar{\varphi}(-y) = -\bar{\varphi}(y),\ \tilde{\varphi}(-y) = -\tilde{\varphi}(y), \\ &\bar{r}(-x) = \tilde{r}(x),\ \tilde{r}(-x) = \bar{r}(x), \\ &\bar{\varphi}(-x) = -\tilde{\varphi}(x),\ \tilde{\varphi}(-x) = -\bar{\varphi}(x). \end{aligned} \qquad \textbf{(S4)}$$

Assuming here the integration over Cartesian variables upon inversion of the variable: $y \to -y$, one obtains: $I^* = \iint_D e^{-il(\bar{\varphi} \pm \tilde{\varphi})} F_l(\bar{r}) F_l(\tilde{r}) dS = I$, which proves real-valuedness of I along with the corresponding matrix elements. Note that here we used the fact that integration area *D* in our examples is always symmetric about the x-axis. For V_{12} the proof is analogous:

$$V_{12}^* = \iint_{SR} e^{2il\bar{\varphi}} F_l^2(\bar{r}) dS \xrightarrow{y \to -y} \iint_{SR} e^{-2il\bar{\varphi}} F_l^2(\bar{r}) dS = V_{12}. \qquad \textbf{(S5)}$$

Since all the matrix elements are real-valued, one can use for them instead of expressions (7) their symmetrized combinations, for example, $V_{ik} \to (V_{ik} + V_{ik}^*)/2$. This would result in replacement of corresponding exponentials by cosine functions, e. g. $e^{il(\tilde{\varphi} - \bar{\varphi})} \to \cos[l(\tilde{\varphi} - \bar{\varphi})]$.

Consider now the properties of matrix elements $\langle i | \hat{l}_z | j \rangle$. According to definition, one has:

$$\begin{aligned} &\langle 1 | \hat{l}_z | 3 \rangle = \iint_D e^{-il\bar{\varphi}} F_l(\bar{r}) \hat{l}_z F_l(\tilde{r}) e^{il\tilde{\varphi}} dS, \\ &\langle 2 | \hat{l}_z | 4 \rangle = \iint_D e^{il\bar{\varphi}} F_l(\bar{r}) \hat{l}_z F_l(\tilde{r}) e^{-il\tilde{\varphi}} dS, \end{aligned} \qquad \textbf{(S6)}$$

where *D* is the total cross-section. Since $\hat{l}_z^* = -\hat{l}_z$ (do not confuse this operation with the Hermitian conjugation), then it is obvious that $\langle 1 | \hat{l}_z | 3 \rangle = -\langle 2 | \hat{l}_z | 4 \rangle^*$. Analogously, one can also show that $\langle 1 | \hat{l}_z | 4 \rangle = -\langle 2 | \hat{l}_z | 3 \rangle^*$. To establish the connection between $\langle 1 | \hat{l}_z | 2 \rangle$ and $\langle 3 | \hat{l}_z | 4 \rangle$ matrix elements starting from their definitions:

$$\begin{aligned} &\langle 1 | \hat{l}_z | 2 \rangle = \iint_D e^{-il\bar{\varphi}} F_l(\bar{r}) \hat{l}_z F_l(\bar{r}) e^{-il\bar{\varphi}} dS, \\ &\langle 3 | \hat{l}_z | 4 \rangle = \iint_D e^{-il\tilde{\varphi}} F_l(\tilde{r}) \hat{l}_z F_l(\tilde{r}) e^{-il\tilde{\varphi}} dS, \end{aligned} \qquad \textbf{(S7)}$$

one has to make in the integral for $\langle 3 | \hat{l}_z | 4 \rangle$ the substitution $x \to -x$. Note that total cross-section's area *D* is symmetric about the y-axis and is invariant under transform $x \to -x$. Since $\hat{l}_z \propto x\nabla_y - y\nabla_x$ then $\hat{l}_z(-x) = -\hat{l}_z(x)$, as well as $\hat{l}_z(-y) = -\hat{l}_z(y)$. Allowing for (A4) and $\hat{l}_z^* = -\hat{l}_z$ this integral becomes:

$$\langle 3 | \hat{l}_z | 4 \rangle = -\iint_D e^{il\bar{\varphi}} F_l(\bar{r}) \hat{l}_z F_l(\bar{r}) e^{il\bar{\varphi}} dS = \langle 1 | \hat{l}_z | 2 \rangle^*. \qquad \textbf{(S8)}$$

Finally, it is possible to establish that all $\langle i | \hat{l}_z | j \rangle$ elements are real-valued. Indeed, making the change of variable $y \to -y$ in (A6) one obtains, for example:

$$\langle 1 | \hat{l}_z | 3 \rangle = -\iint_D e^{il\bar{\varphi}} F_l(\bar{r}) \hat{l}_z F_l(\tilde{r}) e^{-il\tilde{\varphi}} dS = \langle 1 | \hat{l}_z | 3 \rangle^*. \qquad \textbf{(S9)}$$

The same technique can be used for the other elements. Summarizing, the properties of $\langle i | \hat{l}_z | j \rangle$ elements are:

$$\begin{aligned} &\langle 1 | \hat{l}_z | 3 \rangle = -\langle 2 | \hat{l}_z | 4 \rangle^*,\ \langle 1 | \hat{l}_z | 4 \rangle = -\langle 2 | \hat{l}_z | 3 \rangle^*, \\ &\langle 3 | \hat{l}_z | 4 \rangle = \langle 1 | \hat{l}_z | 2 \rangle^*,\ \operatorname{Im}\langle i | \hat{l}_z | j \rangle = 0. \end{aligned} \qquad \textbf{(S10)}$$

For diagonal elements $\langle i | \hat{l}_z | i \rangle$ the procedure is analogous. For example, for element $\langle 1 | \hat{l}_z | 1 \rangle = \iint_D e^{-il\bar{\varphi}} F_l(\bar{r}) \hat{l}_z F_l(\bar{r}) e^{il\bar{\varphi}} dS$ one has:

$$\langle 1 | \hat{l}_z | 1 \rangle^* = -\iint_D e^{il\bar{\varphi}} F_l(\bar{r}) \hat{l}_z F_l(\bar{r}) e^{-il\bar{\varphi}} dS = -\langle 2 | \hat{l}_z | 2 \rangle.$$

The proof of $\langle 3 | \hat{l}_z | 3 \rangle^* = -\langle 4 | \hat{l}_z | 4 \rangle$ property is analogous. In addition,

$$\begin{aligned} \langle 1 | \hat{l}_z | 1 \rangle &= \iint_D e^{-il\bar{\varphi}} F_l(\bar{r}) \hat{l}_z F_l(\bar{r}) e^{il\bar{\varphi}} dS = (x \to -x) \\ &= -\iint_D e^{il\tilde{\varphi}} F_l(\tilde{r}) \hat{l}_z F_l(\tilde{r}) e^{il\tilde{\varphi}} dS = -\langle 4 | \hat{l}_z | 4 \rangle, \end{aligned}$$

and, in the like manner, $\langle 2 | \hat{l}_z | 2 \rangle = -\langle 3 | \hat{l}_z | 3 \rangle$. All diagonal elements are real-valued. Indeed,

$$\begin{aligned} \langle 1 | \hat{l}_z | 1 \rangle^* &= -\iint_D e^{il\bar{\varphi}} F_l(\bar{r}) \hat{l}_z F_l(\bar{r}) e^{-il\bar{\varphi}} dS = (y \to -y) \\ &= \iint_D e^{-il\bar{\varphi}} F_l(\bar{r}) \hat{l}_z F_l(\bar{r}) e^{il\bar{\varphi}} dS = \langle 1 | \hat{l}_z | 1 \rangle. \end{aligned}$$

Summarizing, one obtains that

$$\begin{aligned} &\langle 1 | \hat{l}_z | 1 \rangle = -\langle 2 | \hat{l}_z | 2 \rangle = \langle 3 | \hat{l}_z | 3 \rangle = -\langle 4 | \hat{l}_z | 4 \rangle; \\ &\operatorname{Im}\langle i | \hat{l}_z | i \rangle = 0. \end{aligned} \qquad \textbf{(S11)}$$

Table S1. Reduced perturbation matrix elements $\langle i|V_R|m\rangle/2\beta$ and differences $\tilde{\beta}_m - \tilde{\beta}_i$ between scalar propagation constants

m \ i	-3	-2	-1	0	1	2	3	Analytical expression
-3	17	-29	-256	1667	-1811	1382	-880	$\langle i\|V_R\|m\rangle/2\beta, m^{-1}$
	0	11822	22041	30070	22041	11822	0	$\tilde{\beta}_m - \tilde{\beta}_i, m^{-1}$
-2	-60	223	-422	607	-737	628	-425	$\langle i\|V_R\|m\rangle/2\beta, m^{-1}$
	-11822	0	10222	18251	10222	0	-11822	$\tilde{\beta}_m - \tilde{\beta}_i, m^{-1}$
-1	-99	191	-297	295	-352	299	-201	$\langle i\|V_R\|m\rangle/2\beta, m^{-1}$
	-22041	-10222	0	8029	0	-10222	-22041	$\tilde{\beta}_m - \tilde{\beta}_i, m^{-1}$
0	-93	138	-164	143	-164	138	-93	$\langle i\|V_R\|m\rangle/2\beta, m^{-1}$
	-30070	-18251	-8029	0	-8029	-18251	-30070	$\tilde{\beta}_m - \tilde{\beta}_i, m^{-1}$
1	-201	299	-352	295	-297	191	-99	$\langle i\|V_R\|m\rangle/2\beta, m^{-1}$
	-22041	-10222	0	8029	0	-10222	-22041	$\tilde{\beta}_m - \tilde{\beta}_i, m^{-1}$
2	-425	628	-737	607	-422	223	-60	$\langle i\|V_R\|m\rangle/2\beta, m^{-1}$
	-11822		10222	18251	10222	0	-11822	$\tilde{\beta}_m - \tilde{\beta}_i, m^{-1}$
3	-880	1382	-1811	1677	-256	-29	17	$\langle i\|V_R\|m\rangle/2\beta, m^{-1}$
	0	11822	22041	30070	22041	11822	0	$\tilde{\beta}_m - \tilde{\beta}_i, m^{-1}$