Supplemental Document

Optics Letters

Phase-difference imaging based on FINCH: supplement

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1. The proof process of $d_2 = d_{LC} + 2h\left(n^{-1} - 1\right)$.

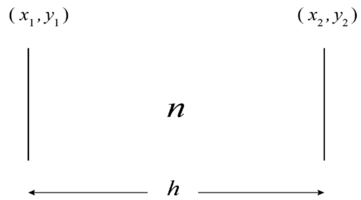

Fig. S1. Schematic diagram

As shown in Fig. S1, the light propagates through a homogenous media with the thickness of h. The light field is $U_1\left(x_1, y_1\right)$ on the x_1y_1 front plane, and it is $U_2\left(x_2, y_2\right)$ on the exit plane. If the medium is air whose refractive index n=1, then the light filed on the x_2y_2 plane is

$$U_2\left(x_2, y_2\right) = \frac{\mathrm{e}^{jkh}}{j\lambda h}\iint U_1\left(x_1, y_1\right) e^{j\frac{k}{2h}\left[(x_2-x_1)^2+(y_2-y_1)^2\right]} dx_1 dy_1, \tag{S1}$$

where λ is the wavelength of light in air. If the light propagates in the medium of refractive index n, the wavelength is replaced with λn^{-1}, and the light filed on the x_2y_2 plane is

$$U_2'\left(x_2, y_2\right) = \frac{\mathrm{e}^{j(kn)h}}{j\left(\lambda n^{-1}\right)h}\iint U_1\left(x_1, y_1\right) e^{j\frac{(kn)}{2h}\left[(x_2-x_1)^2+(y_2-y_1)^2\right]} dx_1 dy_1. \tag{S2}$$

Let $H = hn^{-1}$, then Eq. (S2) can be written as

$$U_2'\left(x_2, y_2\right) = \mathrm{e}^{jk(nh-H)}\frac{\mathrm{e}^{jkH}}{j\lambda H}\iint U_1\left(x_1, y_1\right) e^{j\frac{k}{2H}\left[(x_2-x_1)^2+(y_2-y_1)^2\right]} dx_1 dy_1. \tag{S3}$$

From Eq. (S3), it can be seen that $U_2'\left(x_2, y_2\right)$ is equivalent to $U_1\left(x_1, y_1\right)$ propagating through distance H in air and adding a constant phase $\mathrm{e}^{jk(nh-H)}$. H is rewritten as $H = h + h\left(n^{-1} - 1\right)$. Because the light propagates BS with a thickness of h twice, the equivalent propagation length in air from the lens to the CCD

$$d_2 = d_{LC} + 2h\left(n^{-1} - 1\right), \tag{S4}$$

where d_{LC} is the distance between the lens and the CCD camera.

The additional phase factor $\mathrm{e}^{jk(nh-H)}$ appears in the expression of light field in the above equivalence process. Because this factor appears twice in both the horizontal and vertical components, the products of the horizontal (vertical) component and the conjugate of the vertical (horizontal) component in the interference will cancel out and the four $\mathrm{e}^{jk(nh-H)}$ factors disappear. Therefore, this phase factor will be ignored in the following consideration.

2. The simplified process of $U_\perp(\vec{r}) = \frac{D_1}{d_1 d_2} e^{jk(d_1+d_2)} u_\perp(\vec{r}_o) \times e^{jk\frac{d_1-D_1}{2d_1^2}(x_o^2+y_o^2)} e^{jk\frac{d_2-D_1}{2d_2^2}(x^2+y^2)} e^{-jk\frac{D_1}{d_1 d_2}(xx_o+yy_o)}$ **.**

We assume that the Fresnel approximation condition is always satisfied in our system. The vertical component of the point $\vec{r}_f = (x_f, y_f)$ on the front surface of L is

$$\begin{aligned} P_f(\vec{r}_f) &= u_\perp(\vec{r}_o) d_1^{-1} e^{jkd_1} e^{j\frac{k}{2}[(x_f-x_o)^2+(y_f-y_o)^2]d_1^{-1}} \\ &= u_\perp(\vec{r}_o) d_1^{-1} e^{jkd_1} e^{j\frac{k}{2}(x_o^2+y_o^2)d_1^{-1}} e^{-jk(x_f x_o + y_f y_o)d_1^{-1}} e^{j\frac{k}{2}(x_f^2+y_f^2)d_1^{-1}} \\ &= S_\perp(\vec{r}_o) e^{-jk(x_f x_o + y_f y_o)d_1^{-1}} e^{j\frac{k}{2}(x_f^2+y_f^2)d_1^{-1}}. \end{aligned} \tag{S5}$$

Where $u_\perp(\vec{r}_o)$ is the vertical component of the object point $\vec{r}_o = (x_o, y_o)$, and $S_\perp(\vec{r}_o) = u_\perp(\vec{r}_o) d_1^{-1} e^{jkd_1} e^{j\frac{k}{2}(x_o^2+y_o^2)d_1^{-1}}$. Obviously $S_\perp(\vec{r}_o)$ is independent of the coordinates (x_f, y_f). The light field on the rear surface of lens L is

$$\begin{aligned} U_f(\vec{r}_f) &= P_f(\vec{r}_f) e^{-j\frac{k}{2}(x_f^2+y_f^2)f^{-1}} \\ &= S_\perp(\vec{r}_o) e^{-jk(x_f x_o + y_f y_o)d_1^{-1}} e^{j\frac{k}{2}(x_f^2+y_f^2)d_1^{-1}} e^{-j\frac{k}{2}(x_f^2+y_f^2)f^{-1}} \\ &= S_\perp(\vec{r}_o) e^{-jk(x_f x_o + y_f y_o)d_1^{-1}} e^{j\frac{k}{2}(x_f^2+y_f^2)d_i^{-1}} \end{aligned} \tag{S6}$$

Where *f* is the focal length of lens and $d_i^{-1} = d_1^{-1} - f^{-1}$. The light field of the point $\vec{r} = (x, y)$ on the CCD plane is

$$U_\perp(\vec{r}) = \frac{e^{jkd_2}}{j\lambda d_2} \iint U_f(\vec{r}_f) e^{j\frac{k}{2d_2}[(x-x_f)^2+(y-y_f)^2]} dx_f dy_f. \tag{S7}$$

Eq. (S6) is substituted into Eq. (S7), and the square term in the integral calculation is expanded. Finally, all the terms unrelated to the (x_f, y_f) are extracted to the outside of the integral calculation, and the following result is obtained

$$\begin{aligned} U_\perp(\vec{r}) &= \frac{e^{jkd_2}}{j\lambda d_2} S_\perp(\vec{r}_o) e^{j\frac{k}{2}(x^2+y^2)d_2^{-1}} \\ &\times \iint e^{-jk(x_f x_o + y_f y_o)d_1^{-1}} e^{j\frac{k}{2}(x_f^2+y_f^2)d_i^{-1}} e^{-jk(x_f x + y_f y)d_2^{-1}} e^{j\frac{k}{2}(x_f^2+y_f^2)d_2^{-1}} dx_f dy_f. \end{aligned} \tag{S8}$$

The integral calculation in $U_\perp(\vec{r})$ can be written as the form of Fourier transform, that is

$$\begin{aligned} U_\perp(\vec{r}) &= \frac{e^{jkd_2}}{j\lambda d_2} S_\perp(\vec{r}_o) e^{j\frac{k}{2}(x^2+y^2)d_2^{-1}} \iint e^{j\frac{k}{2}(d_i^{-1}+d_2^{-1})(x_f^2+y_f^2)} \times e^{-j2\pi\left[\left(\frac{x_o}{\lambda d_1}+\frac{x}{\lambda d_2}\right)x_f + \left(\frac{y_o}{\lambda d_1}+\frac{y}{\lambda d_2}\right)y_f\right]} dx_f dy_f \\ &= \frac{e^{jkd_2}}{j\lambda d_2} S_\perp(\vec{r}_o) e^{j\frac{k}{2}(x^2+y^2)d_2^{-1}} \mathcal{F}\left\{e^{-\pi(j\lambda D_1)^{-1}(x_f^2+y_f^2)}\right\}\Big|_{f_x=\frac{x_o}{\lambda d_1}+\frac{x}{\lambda d_2}, f_y=\frac{y_o}{\lambda d_1}+\frac{y}{\lambda d_2}}, \end{aligned} \tag{S9}$$

where $D_1^{-1} = d_i^{-1} + d_2^{-1} = d_1^{-1} + d_2^{-1} - f^{-1}$. By looking at the Fourier transform table, we get

$$\mathcal{F}\left\{e^{-\pi(j\lambda D_1)^{-1}(x_f^2+y_f^2)}\right\}\Big|_{f_x=\frac{x_o}{\lambda d_1}+\frac{x}{\lambda d_2}, f_y=\frac{y_o}{\lambda d_1}+\frac{y}{\lambda d_2}} = j\lambda D_1 e^{-\pi(j\lambda D_1)\left[\left(\frac{x_o}{\lambda d_1}+\frac{x}{\lambda d_2}\right)^2+\left(\frac{y_o}{\lambda d_1}+\frac{y}{\lambda d_2}\right)^2\right]}. \tag{S10}$$

Eq. (S10) and $S_\perp(\vec{r}_o)$ are substituted into Eq. (S9), and obtain

$$U_{\perp}(\vec{r})=\frac{D_1}{d_1d_2}\mathrm{e}^{jk(d_1+d_2)}u_{\perp}(\vec{r}_o)\times e^{jk\frac{d_1-D_1}{2d_1^2}(x_o^2+y_o^2)}e^{jk\frac{d_2-D_1}{2d_2^2}(x^2+y^2)}e^{-jk\frac{D_1}{d_1d_2}(xx_o+yy_o)}. \quad \text{(S11)}$$

3. The simplified process of $U_{//}(\vec{r})=\frac{D_2D_3}{d_1d_3d_4}\mathrm{e}^{jk(d_1+d_3+d_4)}u_{//}(\vec{r}_o)e^{j\frac{k}{2}\left(\frac{d_1-D_2}{d_1^2}-\frac{D_2^2D_3}{d_1^2d_3^2}\right)(x_o^2+y_o^2)}e^{j\frac{k}{2}\frac{d_4-D_3}{d_4^2}(x^2+y^2)}e^{-jk\frac{D_2D_3}{d_1d_3d_4}(xx_o+yy_o)}$.

Eq. (S11) can be used to obtain the light field $U_s(\vec{r}_s)$ of the point $\vec{r}_s=(x_s,y_s)$ on the front surface of SLM in the horizontal direction. Replace $u_{\perp}(\vec{r}_o)$ and d_2 in Eq. (S11) with $u_{//}(\vec{r}_o)$ and d_3, respectively, we can obtain

$$U_S(\vec{r}_s)=\frac{D_2}{d_1d_3}\mathrm{e}^{jk(d_1+d_3)}u_{//}(\vec{r}_o)\times e^{jk\frac{d_1-D_2}{2d_1^2}(x_o^2+y_o^2)}e^{jk\frac{d_3-D_2}{2d_3^2}(x_s^2+y_s^2)}e^{-jk\frac{D_2}{d_1d_3}(x_sx_o+y_sy_o)}. \quad \text{(S12)}$$

Where $D_2^{-1}=d_1^{-1}+d_3^{-1}-f^{-1}$ and $u_{\perp}(\vec{r}_o)$ is the vertical component of the object point $\vec{r}_o=(x_o,y_o)$. Let the focal length of the lens loaded on the SLM be f_s, then the light field on the rear surface of the SLM is $U_{f_s}(\vec{r}_s)=U_S(\vec{r}_s)\times\exp[-j(k/2)(x_s^2+y_s^2)f_s^{-1}]$, and the light field on the CCD surface is

$$U_{//}(\vec{r})=\frac{\mathrm{e}^{jkd_4}}{j\lambda d_4}\iint U_{fs}(\vec{r}_s)e^{j\frac{k}{2d_4}[(x-x_s)^2+(y-y_s)^2]}dx_sdy_s. \quad \text{(S13)}$$

The complete expression of $U_{fs}(\vec{r}_s)$ is substituted into $U_{//}(\vec{r})$, and all terms that are not related to the (x_s, y_s) are extracted out of the integral calculation. Then we get Eq. (S14).

$$\begin{aligned}U_{//}(\vec{r})=&\frac{\mathrm{e}^{jkd_4}}{j\lambda d_4}\frac{D_2}{d_1d_3}\mathrm{e}^{jk(d_1+d_3)}u_{//}(\vec{r}_o)e^{jk\frac{d_1-D_2}{2d_1^2}(x_o^2+y_o^2)}\\&\times\iint e^{jk\frac{d_3-D_2}{2d_3^2}(x_s^2+y_s^2)}e^{-jk\frac{D_2}{d_1d_3}(x_sx_o+y_sy_o)}\times e^{-jk\frac{1}{2f_s}(x_s^2+y_s^2)}e^{jk\frac{1}{2d_4}[(x-x_s)^2+(y-y_s)^2]}dx_sdy_s.\end{aligned} \quad \text{(S14)}$$

To simplify Eq. (S14), let $S_{//}(\vec{r}_o)=D_2(j\lambda d_1d_3d_4)^{-1}\mathrm{e}^{jk(d_1+d_3+d_4)}u_{//}(\vec{r}_o)e^{jk\frac{d_1-D_2}{2d_1^2}(x_o^2+y_o^2)}$. The square terms in the integral calculation are expanded, and the items not related of (x_s, y_s) are extracted out of the integral calculation, and get

$$\begin{aligned}U_{//}(\vec{r})=&S_{//}(\vec{r}_o)e^{j\frac{k}{2}(x^2+y^2)d_4^{-1}}\iint e^{jk\frac{d_3-D_2}{2d_3^2}(x_s^2+y_s^2)}e^{-jk\frac{D_2}{d_1d_3}(x_sx_o+y_sy_o)}\\&\times e^{-jk\frac{1}{2f_s}(x_s^2+y_s^2)}e^{-jk\frac{1}{d_4}(xx_s+yy_s)}e^{jk\frac{1}{2d_4}(x_s^2+y_s^2)}dx_sdy_s.\end{aligned} \quad \text{(S15)}$$

The integral calculation in $U_{//}(\vec{r})$ can be written as the form of Fourier transform, that is

$$\begin{aligned}U_{//}(\vec{r})&=S_{//}(\vec{r}_o)e^{j\frac{k}{2}(x^2+y^2)d_4^{-1}}\iint e^{j\frac{k}{2}\left(\frac{d_3-D_2}{d_3^2}-f_s^{-1}+d_4^{-1}\right)(x_s^2+y_s^2)}e^{-2\pi j\left[\left(\frac{D_2x_o}{\lambda d_1d_3}+\frac{x}{\lambda d_4}\right)x_s+\left(\frac{D_2y_o}{\lambda d_1d_3}+\frac{y}{\lambda d_4}\right)y_s\right]}dx_sdy_s\\&=S_{//}(\vec{r}_o)e^{j\frac{k}{2}(x^2+y^2)d_4^{-1}}\mathcal{F}\left\{e^{-\pi(j\lambda D_3)^{-1}(x_s^2+y_s^2)}\right\}\Big|_{f_x=\frac{D_2x_o}{\lambda d_1d_3}+\frac{x}{\lambda d_4},f_y=\frac{D_2y_o}{\lambda d_1d_3}+\frac{y}{\lambda d_4}},\end{aligned} \quad \text{(S16)}$$

where $D_3^{-1}=d_3^{-1}-D_2d_3^{-2}-f_s^{-1}+d_4^{-1}$. By looking at the Fourier transform table, we get

$$
\mathcal{F}\left\{e^{-\pi(j\lambda D_3)^{-1}\left(x_s^2+y_s^2\right)}\right\}\Big|_{f_x=\frac{D_2x_o}{\lambda d_1d_3}+\frac{x}{\lambda d_4},f_y=\frac{D_2y_o}{\lambda d_1d_3}+\frac{y}{\lambda d_4}} = j\lambda D_3 e^{-\pi(j\lambda D_3)\left[\left(\frac{D_2x_o}{\lambda d_1d_3}+\frac{x}{\lambda d_4}\right)^2+\left(\frac{D_2y_o}{\lambda d_1d_3}+\frac{y}{\lambda d_4}\right)^2\right]}. \tag{S17}
$$

The square term in Eq. (S17) are expanded and substituted into Eq. (S16) together with the expression of $S_{//}\left(\vec{r}_o\right)$. The simplified process is completed.

$$
\begin{aligned}
U_{//}\left(\vec{r}\right)=&\left[\frac{D_2}{j\lambda d_1d_3d_4}\mathrm{e}^{jk(d_1+d_3+d_4)}u_{//}\left(\vec{r}_o\right)e^{jk\frac{d_1-D_2}{2d_1^2}\left(x_o^2+y_o^2\right)}\right]e^{j\frac{k}{2}\left(x^2+y^2\right)d_4^{-1}}\\
&\times\left\{j\lambda D_3 e^{-j\lambda\pi D_3\left[\left(\frac{D_2}{\lambda d_1d_3}\right)^2\left(x_o^2+y_o^2\right)\right]}e^{-j\lambda\pi D_3\left[\left(\frac{1}{\lambda d_4}\right)^2\left(x^2+y^2\right)\right]}e^{-j\lambda\pi D_3\left[\frac{2D_2}{\lambda d_1d_3\lambda d_4}\left(xx_o+yy_o\right)\right]}\right\}\\
=&\frac{D_2D_3}{d_1d_3d_4}\mathrm{e}^{jk(d_1+d_3+d_4)}u_{//}\left(\vec{r}_o\right)e^{j\frac{k}{2}\left(\frac{d_1-D_2}{d_1^2}-\frac{D_2^2D_3}{d_1^2d_3^2}\right)\left(x_o^2+y_o^2\right)}e^{j\frac{k}{2}\frac{d_4-D_3}{d_4^2}\left(x^2+y^2\right)}e^{-jk\frac{D_2D_3}{d_1d_3d_4}\left(xx_o+yy_o\right)}.
\end{aligned} \tag{S18}
$$

4. The simplified process of $U_{//}\left(\vec{r}\right)U_{\perp}^*\left(\vec{r}\right)=-D^{-1}A^{-2}f^2f_su_{//}\left(\vec{r}_o\right)u_{\perp}^*\left(\vec{r}_o\right)e^{j\frac{k}{2D}\left[\left(x-A^{-1}A_ox_o\right)^2+\left(y-A^{-1}A_oy_o\right)^2\right]}$.

Let $U_{//}\left(\vec{r}\right)U_{\perp}^*\left(\vec{r}\right)=U_cU_r$, where U_c is the set of constant coefficients of all items not related to the coordinates r_o and r, and U_r is the set of items related to r_o and r.

$$
U_c=\left(\frac{D_2D_3}{d_1d_3d_4}\mathrm{e}^{jk(d_1+d_3+d_4)}\right)\left(\frac{D_1}{d_1d_2}\mathrm{e}^{jk(d_1+d_2)}\right)^*=C_1, \tag{S19}
$$

$$
\begin{aligned}
U_r=&\left(u_{//}\left(\vec{r}_o\right)e^{j\frac{k}{2}\left(\frac{d_1-D_2}{d_1^2}-\frac{D_2^2D_3}{d_1^2d_3^2}\right)\left(x_o^2+y_o^2\right)}e^{j\frac{k}{2}\frac{d_4-D_3}{d_4^2}\left(x^2+y^2\right)}e^{j\frac{k}{2}\frac{-2D_2D_3}{d_1d_3d_4}\left(xx_o+yy_o\right)}\right)\\
&\times\left(u_{\perp}\left(\vec{r}_o\right)\times e^{j\frac{k}{2}\frac{d_1-D_1}{d_1^2}\left(x_o^2+y_o^2\right)}e^{j\frac{k}{2}\frac{d_2-D_1}{d_2^2}\left(x^2+y^2\right)}e^{j\frac{k}{2}\frac{-2D_1}{d_1d_2}\left(xx_o+yy_o\right)}\right)^*\\
=&u_{//}\left(\vec{r}_o\right)u_{\perp}^*\left(\vec{r}_o\right)\times e^{j\frac{k}{2}C_2\left(x_o^2+y_o^2\right)}e^{j\frac{k}{2}C_3\left(x^2+y^2\right)}e^{j\frac{k}{2}C_4\left(xx_o+yy_o\right)}.
\end{aligned} \tag{S20}
$$

The process of manually simplifying the constant factor C_n (n = 1,2,3,4) in Eq. (S19) and Eq. (S20) is complicated, and it is difficult to find the relationship between them. Therefore, we use Python to process. We set the distances d_1, d_3, d_4 and lens focal length f and f_s as known quantities, and d_2, D_1, D_2, D_3 are represented by the known quantities, that is

$$
d_2=d_3+d_4, \tag{S21}
$$

$$
D_1^{-1}=d_1^{-1}+d_2^{-1}-f^{-1}, \tag{S22}
$$

$$
D_2^{-1}=d_1^{-1}+d_3^{-1}-f^{-1}, \tag{S23}
$$

$$
D_3^{-1}=d_3^{-1}-D_2d_3^{-2}-f_s^{-1}+d_4^{-1}. \tag{S24}
$$

The Python code is at the end of the supplemental document. It is found that every C_n contains the same factor a α. And we get

$$
\begin{aligned}
\alpha=&\left[d_1d_3+d_1d_4-d_1f-d_3f-d_4f\right]\\
&\times\left[d_1d_3d_4-d_1d_3f_s-d_1d_4f-d_1d_4f_s+d_1f_sf-d_3d_4f+d_3f_sf+d_4f_sf\right],
\end{aligned} \tag{S25}
$$

$$
U_c=-\alpha^{-1}f^2f_s, \tag{S26}
$$

$$U_r = u_{//}\left(\vec{r}_o\right)u_\perp^*\left(\vec{r}_o\right)e^{j\frac{k}{2}\alpha^{-1}\left(d_4 f\right)^2\left(x_o^2+y_o^2\right)}e^{j\frac{k}{2}\alpha^{-1}\left(d_1d_3-d_1f-d_3f\right)^2\left(x^2+y^2\right)} \times e^{-j\frac{k}{2}\alpha^{-1}\times 2\left(d_4 f\right)\left(d_1d_3-d_1f-d_3f\right)\left(xx_o+yy_o\right)}. \tag{S27}$$

Let $A_o = d_4 f$, $A = d_1d_3 - d_1 f - d_3 f$,α and U_r are expressed as

$$\alpha = \left[d_1d_2 - \left(d_1+d_2\right)f\right]\times\left\{\left[\left(d_1+d_2\right)f - d_1d_2\right]f_s + Ad_4\right\}, \tag{S28}$$

$$U_r = u_{//}\left(\vec{r}_o\right)u_\perp^*\left(\vec{r}_o\right)e^{j\frac{k}{2}\alpha^{-1}\left[A_o^2\left(x_o^2+y_o^2\right)\right]}e^{j\frac{k}{2}\alpha^{-1}\left[A^2\left(x^2+y^2\right)\right]}e^{-j\frac{k}{2}\alpha^{-1}\left[2A_oA\left(xx_o+yy_o\right)\right]}. \tag{S29}$$

The exponential part of U_r can be written as square form. And we get

$$U_{//}\left(\vec{r}\right)U_\perp^*\left(\vec{r}\right) = -\alpha^{-1}f^2 f_s u_{//}\left(\vec{r}_o\right)u_\perp^*\left(\vec{r}_o\right)e^{j\frac{k}{2}\left(\alpha^{-1}\right)\left[\left(Ax-A_ox_o\right)^2+\left(Ay-A_oy_o\right)^2\right]}. \tag{S30}$$

Let $D = \alpha A^{-2}$, Eq. (S30) can be rewritten as Eq. (S31), and the simplified process is completed.

$$U_{//}\left(\vec{r}\right)U_\perp^*\left(\vec{r}\right) = -D^{-1}A^{-2}f^2 f_s u_{//}\left(\vec{r}_o\right)u_\perp^*\left(\vec{r}_o\right)e^{j\frac{k}{2D}\left[\left(x-A^{-1}A_ox_o\right)^2+\left(y-A^{-1}A_oy_o\right)^2\right]}. \tag{S31}$$

Python code for the simplified process of C_n

```
from sympy import symbols, exp, factor
d1, d3, d4, f, fs, k = symbols('d1 d3 d4 f fs k')
d2 = d3 + d4
D1 = 1 / (1 / d1 + 1 / d2 - 1 / f)
D2 = 1 / (1 / d1 + 1 / d3 - 1 / f)
D3 = 1 / (1 / d3 - D2 / d3 ** 2 - 1 / fs + 1 / d4)
C1 = (D2 * D3 / d1 / d3 / d4 * exp(1j * k * (d1 + d3 + d4))) * (D1 / d1 / d2 * exp(-1j * k * (d1 + d2)))
C2 = (d1 - D2) / d1 ** 2 - D2 ** 2 * D3 / (d1 ** 2 * d3 ** 2) - (d1 - D1) / d1 ** 2
C3 = (d4 - D3) / d4 ** 2 - (d2 - D1) / d2 ** 2
C4 = -2 * D2 * D3 / d1 / d3 / d4 - (-2 * D1 / d1 / d2)
C1 = factor(C1)
C2 = factor(C2)
C3 = factor(C3)
C4 = factor(C4)
print(f'C1 = {C1}')
print(f'C2 = {C2}')
print(f'C3 = {C3}')
print(f'C4 = {C4}')
```