Supplemental Document

Optics EXPRESS

Ultranarrow spectral line of the radiation in double qubit-cavity ultrastrong coupling system: supplement

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Supplement DOI: https://doi.org/10.6084/m9.figshare.14912832

Parent Article DOI: https://doi.org/10.1364/OE.432358

1. EQUATIONS OF MOTION AND STATIONARY SOLUTIONS

The equations of motion for the density-matrix elements of the system can be expressed as

$$
\begin{aligned}
\dot{\rho}_{00} =& \Gamma_{10}\rho_{11} + \Gamma_{20}\rho_{22} + \Gamma_{30}\rho_{33} + \frac{i\,\Omega}{2}(\rho_{03} - \rho_{30}),\\
\dot{\rho}_{11} =& -\Gamma_{10}\rho_{11} + +\Gamma_{21}\rho_{22} + \Gamma_{31}\rho_{33},\\
\dot{\rho}_{33} =& -\frac{\Gamma_{30}+\Gamma_{31}+\Gamma_{32}}{2}\rho_{33} - \frac{i\,\Omega}{2}(\rho_{03} - \rho_{30}),\\
\dot{\rho}_{03} =& -\frac{\Gamma_{30}+\Gamma_{31}+\Gamma_{32}}{2}\rho_{03} + \frac{i\,\Omega}{2}(\rho_{00} - \rho_{33}),\\
\dot{\rho}_{01} =& -\frac{\Gamma_{10}}{2}\rho_{01} - \frac{i\,\Omega}{2}\rho_{31},\\
\dot{\rho}_{13} =& -\frac{\Gamma_{10}+\Gamma_{30}+\Gamma_{31}+\Gamma_{32}}{2}\rho_{13} + \frac{i\,\Omega}{2}\rho_{10},
\end{aligned}
\tag{S1}
$$

with $\rho_{22} = 1 - \rho_{00} - \rho_{11} - \rho_{33}$ and $\rho_{ji} = (\rho_{ij})^*$ for $i, j \in \{1, 2, 3\}$. Introducing $A = \Gamma_{30} + \Gamma_{31} + \Gamma_{32}$ and $B = (2\Gamma_{10} + \Gamma_{31})(\Gamma_{20} + \Gamma_{21}) + \Gamma_{32}(\Gamma_{10} + \Gamma_{21})$, the stationary solutions of the system can be derived as

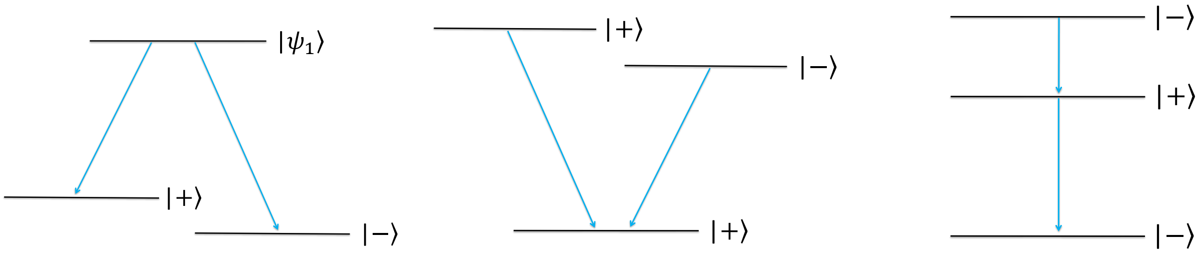
$$
\begin{aligned}
\rho_{00}^{ss} =& \frac{\Gamma_{10}(\Gamma_{20}+\Gamma_{21})(A^2+\Omega^2)}{\Gamma_{10}(\Gamma_{20}+\Gamma_{21})A^2 + B\,\Omega^2},\\
\rho_{11}^{ss} =& \frac{[\Gamma_{31}(\Gamma_{20}+\Gamma_{21}) + \Gamma_{32}\Gamma_{21}]\,\Omega^2}{\Gamma_{10}(\Gamma_{20}+\Gamma_{21})A^2 + B\,\Omega^2},\\
\rho_{22}^{ss} =& \frac{\Gamma_{10}\Gamma_{32}\Omega^2}{\Gamma_{10}(\Gamma_{20}+\Gamma_{21})A^2 + B\,\Omega^2},\\
\rho_{33}^{ss} =& \frac{\Gamma_{10}(\Gamma_{20}+\Gamma_{21})\Omega^2}{\Gamma_{10}(\Gamma_{20}+\Gamma_{21})A^2 + B\,\Omega^2},\\
\rho_{03}^{ss} =& \frac{i\Gamma_{10}(\Gamma_{20}+\Gamma_{21})A\Omega}{\Gamma_{10}(\Gamma_{20}+\Gamma_{21})A^2 + B\,\Omega^2}, \quad \rho_{01}^{ss} = \rho_{13}^{ss} = 0.
\end{aligned}
\tag{S2}
$$

With these analytical expressions in hand, we can obtain steady-state solutions for different coupling strengths g. For instance, when we set the parameters as $\varepsilon/\omega_q = 8 \times 10^{-3}$, $\omega_c/\omega_q = 1.915$, $\kappa/\omega_q = 2 \times 10^{-3}$, $\gamma/\omega_q = 2 \times 10^{-5}$, and $\theta = \pi/6$. We have $\rho_{00}^{ss} = 0.34, \rho_{11}^{ss} = 0.24, \rho_{22}^{ss} = 0.09, \rho_{33}^{ss} = 0.33$ for $g/\omega_q = 0.2$, and $\rho_{00}^{ss} = 0.22, \rho_{11}^{ss} = 0.56, \rho_{22}^{ss} = 0, \rho_{33}^{ss} = 0.22$ for $g/\omega_q = 0.7056$. As can be seen from Eq. (S2), $\rho_{00}^{ss} \approx \rho_{33}^{ss}$ for the condition that $\Omega^2 \gg A^2$. With the increase of g from $0.2\omega_q$ to $0.7056\omega_q$, Γ_{10} and Γ_{20} are almost unchanged, while Γ_{31} increases significantly and Γ_{32} decreases to zero. It means that the transition process $|\psi_3\rangle \to |\psi_1\rangle$ is enhanced but the transition probability of $|\psi_1\rangle \to |\psi_0\rangle$ stays the same, which leads to the accumulation of population ρ_{11}^{ss}. In addition, $\Gamma_{32} = 0$ makes the transition from $|\psi_3\rangle$ to $|\psi_2\rangle$ almost quenched, which resulting in $\rho_{22}^{ss} \simeq 0$.

Furthermore, in the representation of dressed state that $|\pm\rangle = (|\psi_3\rangle \pm |\psi_0\rangle)/\sqrt{2}$, the master equation of the reduced density operator ρ takes the form

$$
\begin{aligned}
\dot{\rho} =& -i[H_s, \rho] + \frac{\Gamma_{10}}{4}\Big[(2\sigma_{+1}\rho\sigma_{1+} - \sigma_{11}\rho - \rho\sigma_{11}) + (2\sigma_{-1}\rho\sigma_{1-} - \sigma_{11}\rho - \rho\sigma_{11}) - (2\sigma_{+1}\rho\sigma_{1-} + h.c.)\Big] + \frac{\Gamma_{20}}{4}\Big[(2\sigma_{+2}\rho\sigma_{2+} - \sigma_{22}\rho\\
& - \rho\sigma_{22}) + (2\sigma_{-2}\rho\sigma_{2-} - \sigma_{22}\rho - \rho\sigma_{22}) - (2\sigma_{+2}\rho\sigma_{2-} + h.c.)\Big] + \frac{\Gamma_{30}}{8}\Big[(2\sigma_{++}\rho\sigma_{++} - \sigma_{++}\rho - \rho\sigma_{++}) + (2\sigma_{--}\rho\sigma_{--} - \sigma_{--}\rho - \rho\sigma_{--})\\
& + (2\sigma_{+-}\rho\sigma_{-+} - \sigma_{--}\rho - \rho\sigma_{--}) + (2\sigma_{-+}\rho\sigma_{+-} - \sigma_{++}\rho - \rho\sigma_{++}) - (2\sigma_{++}\rho\sigma_{--} + h.c.) - (2\sigma_{+-}\rho\sigma_{+-} + h.c.) + (2\sigma_{++}\rho\sigma_{-+}\\
& - \sigma_{-+}\rho - \rho\sigma_{-+} + h.c.) + (2\sigma_{--}\rho\sigma_{+-} - \sigma_{+-}\rho - \rho\sigma_{+-} + h.c.) - (2\sigma_{++}\rho\sigma_{+-} + 2\sigma_{--}\rho\sigma_{-+} + h.c.)\Big] + \frac{\Gamma_{31}}{4}\Big[(2\sigma_{1+}\rho\sigma_{+1} - \sigma_{++}\rho\\
& - \rho\sigma_{++}) + (2\sigma_{1-}\rho\sigma_{-1} - \sigma_{--}\rho - \rho\sigma_{--}) + (2\sigma_{1+}\rho\sigma_{-1} - \sigma_{-+}\rho - \rho\sigma_{-+} + h.c.)\Big] + \frac{\Gamma_{32}}{4}\Big[(2\sigma_{2+}\rho\sigma_{+2} - \sigma_{++}\rho - \rho\sigma_{++})\\
& + (2\sigma_{2-}\rho\sigma_{-2} - \sigma_{--}\rho - \rho\sigma_{--}) + (2\sigma_{2+}\rho\sigma_{-2} - \sigma_{-+}\rho - \rho\sigma_{-+} + h.c.)\Big] + \frac{\Gamma_{21}}{4}\Big[(2\sigma_{12}\rho\sigma_{21} - \sigma_{22}\rho - \rho\sigma_{22})\Big].
\end{aligned}
\tag{S3}
$$

In this master equation, in addition to the conventional damping between the dressed states, such as $|\pm\rangle \to |\pm\rangle$, $|\pm\rangle \to |\mp\rangle$, $|\pm\rangle \to |\psi_{1,2}\rangle$, and $|\psi_2\rangle \to |\psi_1\rangle$, there are some cross coupling terms that reflect the quantum interference between two transition pathways. For example, $\sigma_{+1}\rho\sigma_{1-}$ represents the quantum interference between transition channels $|\psi_1\rangle \to |+\rangle$ and $|\psi_1\rangle \to |-\rangle$. In order to clarify the origin of the cross coupling term, we use the three-level Λ-type model to reconstruct the energy level structure of dressed states $|\psi_1\rangle$, $|+\rangle$, and $|-\rangle$, as shown in Fig. S1(a). Analogously, we can generalize the origin of all the cross coupling terms in the master equation to the quantum interference between two dissipative channels in the Λ-type [1, 2], V-type [3, 4], and Ξ-type [5, 6] three-level structures, as shown in Fig. S1.



(a) Λ-type (b) V − type (c) Ξ − type

Fig. S1. The three-level schemes for the cross coupling terms (a) $\sigma_{+1}\rho\sigma_{1-}$, (b) $2\sigma_{++}\rho\sigma_{-+} - \sigma_{-+}\rho - \rho\sigma_{-+}$, and (c) $\sigma_{+-}\rho\sigma_{+-}$. Replace the upper level $|\psi_1\rangle$ in Fig. S1(a) with $|\psi_2\rangle$, $|+\rangle$, and $|-\rangle$ to represent the cross coupling term $\sigma_{+2}\rho\sigma_{2-}$, $\sigma_{++}\rho\sigma_{+-}$, and $\sigma_{--}\rho\sigma_{-+}$ in the master equation, respectively. And Replace the lower level $|+\rangle$ in Fig. S1(b) with $|-\rangle$, $|\psi_1\rangle$, and $|\psi_2\rangle$ to represent the cross coupling term $2\sigma_{--}\rho\sigma_{+-} - \sigma_{+-}\rho - \rho\sigma_{+-}$, $2\sigma_{1+}\rho\sigma_{-1} - \sigma_{-+}\rho - \rho\sigma_{-+}$, and $2\sigma_{2+}\rho\sigma_{-2} - \sigma_{-+}\rho - \rho\sigma_{-+}$ in the master equation.

From Eq. (S3), we can obtain the equations of motion for the density matrix elements as following

$$\frac{d}{dt}\vec{\rho}_i = \mathbf{M}_i\vec{\rho}_i + \mathbf{I}_i, \tag{S4}$$

here we define vectors $\vec{\rho}_1 = (\rho_{+-}, \rho_{-+}, \rho_{++}, \rho_{--}, \rho_{11})^T$ and $\vec{\rho}_2 = (\rho_{+1}, \rho_{-1})^T$. The matrices of coefficients $\mathbf{M}_i$ and the constant vectors $\mathbf{I}_i$ are given by

$$\mathbf{M}_1 = \begin{pmatrix} -\frac{2A+\Gamma_{30}}{4} - i\Omega & -\frac{\Gamma_{30}}{4} & -\frac{A+\Gamma_{30}}{4} + \frac{\Gamma_{20}}{2} & -\frac{A+\Gamma_{30}}{4} + \frac{\Gamma_{20}}{2} & -\frac{\Gamma_{10}}{2} + \frac{\Gamma_{20}}{2} \\ -\frac{\Gamma_{30}}{4} & -\frac{2A+\Gamma_{30}}{4} + i\Omega & -\frac{A+\Gamma_{30}}{4} + \frac{\Gamma_{20}}{2} & -\frac{A+\Gamma_{30}}{4} + \frac{\Gamma_{20}}{2} & -\frac{\Gamma_{10}}{2} + \frac{\Gamma_{20}}{2} \\ -\frac{\Gamma_{31}+\Gamma_{32}}{4} & -\frac{\Gamma_{31}+\Gamma_{32}}{4} & -\frac{A+\Gamma_{31}+\Gamma_{32}}{4} - \frac{\Gamma_{20}}{2} & \frac{\Gamma_{30}}{4} - \frac{\Gamma_{20}}{2} & \frac{\Gamma_{10}}{2} - \frac{\Gamma_{20}}{2} \\ -\frac{\Gamma_{31}+\Gamma_{32}}{4} & -\frac{\Gamma_{31}+\Gamma_{32}}{4} & \frac{\Gamma_{30}}{4} - \frac{\Gamma_{20}}{2} & -\frac{A+\Gamma_{31}+\Gamma_{32}}{4} - \frac{\Gamma_{20}}{2} & \frac{\Gamma_{10}}{2} - \frac{\Gamma_{20}}{2} \\ \frac{\Gamma_{31}}{2} & \frac{\Gamma_{31}}{2} & \frac{\Gamma_{31}}{2} - \Gamma_{21} & \frac{\Gamma_{31}}{2} - \Gamma_{21} & -\Gamma_{10} - \Gamma_{21} \end{pmatrix}, \tag{S5a}$$

$$\mathbf{M}_2 = \begin{pmatrix} -\frac{A+2\Gamma_{10}}{4} - \frac{i\Omega}{2} & -\frac{A}{4} \\ -\frac{A}{4} & -\frac{A+2\Gamma_{10}}{4} + \frac{i\Omega}{2} \end{pmatrix}, \tag{S5b}$$

and $\mathbf{I}_1 = (-\frac{\Gamma_{20}}{2}, -\frac{\Gamma_{20}}{2}, \frac{\Gamma_{20}}{2}, \frac{\Gamma_{20}}{2}, \Gamma_{21})^T$, $\mathbf{I}_2 = (0, 0)^T$.

According to the motion equations in Eq. (S4), the stationary solution of the density matrix elements in the dressed state representation can be derived as

$$\begin{aligned} \rho_{++}^{ss} &= \frac{(1+\xi^2)/2}{1+\xi^2+(\beta_+/2)\xi^2}, \quad \rho_{--}^{ss} = \rho_{++}^{ss}, \\ \rho_{+-}^{ss} &= \frac{-(1-\sqrt{2}i\xi)/2}{1+\xi^2+(\beta_+/2)\xi^2}, \quad \rho_{-+}^{ss} = (\rho_{+-}^{ss})^*, \\ \rho_{11}^{ss} &= \frac{(\beta_+ + \beta_-)\xi^2/4}{1+\xi^2+(\beta_+/2)\xi^2}, \quad \rho_{22}^{ss} = \frac{(\beta_+ - \beta_-)\xi^2/4}{1+\xi^2+(\beta_+/2)\xi^2}, \end{aligned} \tag{S6}$$

where

$$\beta_\pm = \frac{\Gamma_{31}}{\Gamma_{10}} + \frac{\Gamma_{32}(\Gamma_{21} \pm \Gamma_{10})}{\Gamma_{10}(\Gamma_{20}+\Gamma_{21})}, \quad \xi = \frac{\sqrt{2}\Omega}{A}. \tag{S7}$$

2. ANALYTICAL RESULTS OF CAVITY EMISSION SPECTRUM

A. DERIVATION OF BROAD PEAKS

For the major component of the emission spectrum $S_1(\omega)$, we can transform $\langle\delta\sigma_{30}(0)\delta\sigma_{03}(\tau)\rangle$ into the dressed state representation, and divide it into the following two parts

$$\langle\delta\sigma_{30}(0)\delta\sigma_{03}(\tau)\rangle = \frac{1}{2}\Big[\langle\delta\sigma_{30}(0)\left(\delta\sigma_{++}(\tau)-\delta\sigma_{--}(\tau)\right)\rangle + \langle\delta\sigma_{30}(0)\left(\delta\sigma_{+-}(\tau)-\delta\sigma_{-+}(\tau)\right)\rangle\Big], \tag{S8}$$

where $\sigma_{30} = (\sigma_{++}-\sigma_{--}+\sigma_{-+}-\sigma_{+-})/2$. Generally, the first part of Eq. (S8) corresponds to the central peak of the spectrum. And the last part is associated with different frequencies, which corresponds to the sidebands of the Mollow-like spectrum.

Since the motion equations Eq. (S4) is too complicated, here we adopt the secular approximation [7] to derive the five broad peaks of the fluorescence spectrum. When $\Omega \gg \Gamma_{mn}$, the contributions from terms with different frequencies in the motion equations are negligibly. The equations of motion in the dressed state representation are given by

$$\dot{\rho}_{++}-\dot{\rho}_{--} = -\frac{A}{2}\left(\rho_{++}-\rho_{--}\right), \tag{S9a}$$

$$\dot{\rho}_{+-} \simeq -\left(\frac{2A+\Gamma_{30}}{4}+i\Omega\right)\rho_{+-}. \tag{S9b}$$

According to the quantum regression theorem, when $\tau > 0$, the analytical form of the Mollow spectrum can be derived as as

$$S_1'(\omega) = \frac{|\alpha_{03}|^2}{2}\mathcal{R}\left[\frac{C_0}{\lambda_0-i\omega}+\frac{C_0^+}{\lambda_0^+-i\omega}+\frac{C_0^-}{\lambda_0^--i\omega}\right], \tag{S10}$$

where

$$\lambda_0 = \frac{A}{2}, \qquad C_0 = \rho_{++}^{ss}+\rho_{--}^{ss},$$

$$\lambda_0^\pm = \frac{2A+\Gamma_{30}}{4}\pm i\Omega, \quad C_0^\pm = \rho_{\pm\pm}^{ss}. \tag{S11}$$

From Eq. (S11), we can see that the linewidth of the central peak is $2\lambda_0$ and the height is $|\alpha_{03}|^2C_0/2\lambda_0$. The linewidth of the outer sidebands which located at $\pm\Omega$ is $(2A+\Gamma_{30})/2$.

Analogously, for the other two components of the emission spectrum $S_2(\omega)$ and $S_3(\omega)$, we have

$$\langle\delta\sigma_{10}(0)\delta\sigma_{01}(\tau)\rangle = \frac{1}{2}\Big[\langle\delta\sigma_{1+}(0)\delta\sigma_{+1}(\tau)\rangle+\langle\delta\sigma_{1-}(0)\delta\sigma_{-1}(\tau)\rangle-\langle\delta\sigma_{1+}(0)\delta\sigma_{-1}(\tau)\rangle-\langle\delta\sigma_{1-}(0)\delta\sigma_{+1}(\tau)\rangle\Big], \tag{S12a}$$

$$\langle\delta\sigma_{31}(0)\delta\sigma_{13}(\tau)\rangle = \frac{1}{2}\Big[\langle\delta\sigma_{+1}(0)\delta\sigma_{1+}(\tau)\rangle+\langle\delta\sigma_{-1}(0)\delta\sigma_{1-}(\tau)\rangle+\langle\delta\sigma_{+1}(0)\delta\sigma_{1-}(\tau)\rangle+\langle\delta\sigma_{-1}(0)\delta\sigma_{1+}(\tau)\rangle\Big]. \tag{S12b}$$

The equations of motion under secular approximation are given by

$$\dot{\rho}_{1+} \simeq -\left(\frac{A+2\Gamma_{10}}{4}-\frac{i\Omega}{2}\right)\rho_{1+}, \tag{S13a}$$

$$\dot{\rho}_{1-} \simeq -\left(\frac{A+2\Gamma_{10}}{4}+\frac{i\Omega}{2}\right)\rho_{1-}. \tag{S13b}$$

Then we can get the analytical form of the additional sidebands as

$$S_2(\omega) = |\alpha_{01}|^2\mathcal{R}\left[\frac{\rho_{11}^{ss}}{\lambda_2^+-i\omega}+\frac{\rho_{11}^{ss}}{\lambda_2^--i\omega}\right], \tag{S14a}$$

$$S_3(\omega) = |\alpha_{13}|^2\mathcal{R}\left[\frac{\rho_{++}^{ss}}{\lambda_2^+-i\omega}+\frac{\rho_{--}^{ss}}{\lambda_2^--i\omega}\right], \tag{S14b}$$

where

$$\lambda_2^\pm = \frac{A+2\Gamma_{10}}{4}\pm\frac{i\Omega}{2}, \tag{S15}$$

Obviously, the contributions of $S_2(\omega)$ and $S_3(\omega)$ shape the inner sidebands (located at $\pm\Omega/2$) of the spectrum.

In summary, the linewidths of the five peaks obtained above all depend on A, which contains the relaxation coefficient Γ_{30} from $|\psi_3\rangle$ to $|\psi_0\rangle$. That is a relatively strong damping process, so these five peaks are all broad peaks.

B. DERIVATION OF ULTRANARROW PEAK

So far, we have derived the analytical expressions for the components of the spectrum corresponding to the Mollow-like triplet and the extra sidebands. However, through the method of secular approximation, we can not find the component of the narrow peak imposed on the central peak, which is the most significant feature we care about. Therefore, we doubt that we lost some small quantities that should not be ignored via secular approximation approach. In order to bring it back, we refocused on Eq. (S9). The dynamic evolution equation of the population difference $\rho_{++} - \rho_{--}$ in Eq. (S9a) is an exact equation that directly obtained through the equations of motion for $\vec{\rho}_1$, and no secular approximation is used here. Therefore, when detecting the fluorescence radiated by the transition pathways $|\pm\rangle \to |\pm\rangle$, which corresponds to observation operator $\langle\delta\sigma_{++} - \delta\sigma_{--}\rangle$, we can only observe the central peak of the Mollow-like triplet, and we cannot see the narrow peak. In order to dig out the origin of the narrow peak, we can only start with the fluorescence observation operator $\langle\delta\sigma_{+-} - \delta\sigma_{-+}\rangle$ for the sidebands. First we rewrite Eq. (S9b) without secular approximation

$$\dot{\rho}_{+-} = -\left(\frac{2A+\Gamma_{30}}{4} + i\Omega\right)\rho_{+-} - \frac{\Gamma_{30}}{4}\rho_{-+} - \frac{A+\Gamma_{30}}{4}\left(\rho_{++}+\rho_{--}\right) - \frac{\Gamma_{10}}{2}\rho_{11} - \frac{\Gamma_{20}}{2}\rho_{22}. \tag{S16}$$

From this equation, we can see that the terms that ignored by secular approximation are all introduced through the cross coupling of two dissipative channels, as shown in Fig. S1. As we mentioned in the following paragraph of Eq. (S3), quantum interference between two dissipative channels establishes the coupling of terms with different frequency in the dynamic evolution equation of ρ_{+-}, laying the foundation for the emergence of narrow peak. For instance, the $\rho_{++} + \rho_{--}$ in Eq. (S16) comes from the quantum interference of two transition channels in the V-type three-level structure formed by the operators $2\sigma_{++}\rho\sigma_{-+} - \sigma_{-+}\rho - \rho\sigma_{-+}$, $2\sigma_{--}\rho\sigma_{+-} - \sigma_{+-}\rho - \rho\sigma_{+-}$, $2\sigma_{1+}\rho\sigma_{-1} - \sigma_{-+}\rho - \rho\sigma_{-+}$, and $2\sigma_{2+}\rho\sigma_{-2} - \sigma_{-+}\rho - \rho\sigma_{-+}$.

In order to figure out the two-time correlation of the last part in Eq. (S8), we rewrite its dynamic evolution equation and the coupled equations. The results are as follows

$$\frac{d}{dt}\left(\rho_{-+} - \rho_{+-}\right) = -\frac{A}{2}\left(\rho_{-+} - \rho_{+-}\right) + i\Omega\left(\rho_{-+} + \rho_{+-}\right), \tag{S17a}$$

$$\frac{d}{dt}\left(\rho_{-+} + \rho_{+-}\right) = -\frac{A+\Gamma_{30}}{2}\left(\rho_{-+} + \rho_{+-}\right) + i\Omega\left(\rho_{-+} - \rho_{+-}\right) + \left(\Gamma_{12}^{+} - \frac{A+\Gamma_{30}}{2}\right)\left(\rho_{++} + \rho_{--}\right) - \Gamma_{12}^{-}\left(\rho_{11} - \rho_{22}\right), \tag{S17b}$$

$$\frac{d}{dt}\left(\rho_{++} + \rho_{--}\right) = -\left(\Gamma_{12}^{+} + \Gamma_{23}^{+}\right)\left(\rho_{++} + \rho_{--}\right) - \Gamma_{23}^{+}\left(\rho_{-+} + \rho_{+-}\right) + \Gamma_{12}^{-}\left(\rho_{11} - \rho_{22}\right), \tag{S17c}$$

$$\frac{d}{dt}\left(\rho_{11} - \rho_{22}\right) = -\left(\Gamma_{12}^{+} + \Gamma_{21}\right)\left(\rho_{11} - \rho_{22}\right) + \Gamma_{23}^{-}\left(\rho_{-+} + \rho_{+-}\right) + \left(\Gamma_{12}^{-} + \Gamma_{23}^{-} - \Gamma_{21}\right)\left(\rho_{++} + \rho_{--}\right), \tag{S17d}$$

where $\Gamma_{12}^{\pm} = (\Gamma_{10} \pm \Gamma_{20})/2$ and $\Gamma_{23}^{\pm} = (\Gamma_{31} \pm \Gamma_{32})/2$.

It can be seen from the above equations of motion that due to $A \gg \Gamma_{12}^{+}, \Gamma_{23}^{+}, \Gamma_{21}$, the decay rates $A/2$ and $(A+\Gamma_{30})/2$ corresponding to the first two terms $\rho_{-+} - \rho_{+-}$ and $\rho_{-+} + \rho_{+-}$ are much faster than the decay rates $\Gamma_{12}^{+} + \Gamma_{23}^{+}$ and $\Gamma_{12}^{+} + \Gamma_{21}$ corresponding to the last two terms $\rho_{++} + \rho_{--}$ and $\rho_{11} - \rho_{22}$. That is to say, in the time scale $t = 1/(\Gamma_{12}^{+} + \Gamma_{23}^{+})$ or $1/(\Gamma_{12}^{+} + \Gamma_{21})$, we can assume that the fast decay terms $\rho_{-+} - \rho_{+-}$ and $\rho_{-+} + \rho_{+-}$ have reached the steady state at this time, the so-called time evolution of them is determined by the time evolution of the slow decay terms $\rho_{++} + \rho_{--}$ and $\rho_{11} - \rho_{22}$. Thus we can set the time evolutions of the first two terms equal to 0, and substitute them into the last two equations. Combining with the steady-state solution in Eq. (S6), we find that the initial values of non-zero in $\langle\delta\sigma_{30}(0)\left(\delta\sigma_{+-}(\tau) - \delta\sigma_{-+}(\tau)\right)\rangle$ are only

$$\langle\left(\delta\sigma_{+-}(0) - \delta\sigma_{-+}(0)\right)\left(\delta\sigma_{++}(0) + \delta\sigma_{--}(0)\right)\rangle = (\rho_{11}^{ss} + \rho_{22}^{ss})(\rho_{+-}^{ss} - \rho_{-+}^{ss}), \tag{S18a}$$

$$\langle\left(\delta\sigma_{+-}(0) - \delta\sigma_{-+}(0)\right)\left(\delta\sigma_{11}(0) - \delta\sigma_{22}(0)\right)\rangle = (\rho_{11}^{ss} - \rho_{22}^{ss})(\rho_{-+}^{ss} - \rho_{+-}^{ss}), \tag{S18b}$$

which corresponds to the dynamic evolutions of $\rho_{++} + \rho_{--}$ and $\rho_{11} - \rho_{22}$ in Eqs. (S17c) and (S17d). Therefore, the origin of the narrow peak that located at the center of the spectrum is the correlation between the central peak and the side peaks of Mollow-like triplet. Comparing Eq. (S17c) with Eq. (S9a), we find that the time evolution of the population difference $\rho_{++} - \rho_{--}$ corresponding to the observation operator of the central peak is a rapid decay process with the decay rate $A/2$, while the incoherent injection $\rho_{++} + \rho_{--}$ corresponding to the narrow peak is a slow decay process with the decay rate $\Gamma_{12}^{+} + \Gamma_{23}^{+}$. If there is no electron shelving of the intermediate energy levels $|\psi_1\rangle$ and $|\psi_2\rangle$, such as a two-level atomic system, then the incoherent injection $\rho_{++} + \rho_{--}$ will be a constant that does not evolve with time, and the narrow peak will not appear.

Furthermore, the narrow peak here is different from the narrow peak of the V-type three-level atomic systems [7, 8]. In the V-type three-level atomic system, the narrow peak at the line center is directly derived from the motion equation of the observation operator of the central peak. However, here we find a narrow peak imposed on the central peak when detecting the fluorescence spectrum of the sidebands. Specifically, the vacuum-induced quantum interference coupled the equations Eqs. (S17c) and (S17d) for the central peak with the equations Eqs. (S17a) and (S17b) for the sidebands. This finding provides a possibility for the exploration of the quantum interference effect in the USC system.

Now that we can obtain the analytical form of the narrow peak as

$$S_1''(\omega) = \frac{|\alpha_{03}|^2}{2}\mathcal{R}\left[\frac{C_1^{+}}{\lambda_1^{+} - i\omega} + \frac{C_1^{-}}{\lambda_1^{-} - i\omega}\right], \tag{S19}$$

where

$$\lambda_1^{\pm} = \frac{D \pm \Delta}{4}, \tag{S20a}$$

$$C_1^{\pm} = \frac{i\Gamma_{30}(\rho_{-+}^{ss} - \rho_{+-}^{ss})}{2\Omega} \left[\left(1 \pm \frac{\Gamma_+ + \Gamma_{32}}{\Delta}\right) \rho_{11}^{ss} + \left(1 \pm \frac{\Gamma_- + \Gamma_{32}}{\Delta}\right) \rho_{22}^{ss} \right], \tag{S20b}$$

and $\Delta = \sqrt{\Gamma_+^2 + 2\Gamma_-\Gamma_{32} + \Gamma_{32}^2}$, and $\Gamma_\pm = \Gamma_{31} - \Gamma_{21} \pm 2(\Gamma_{10} - \Gamma_{20})$, $D = 2\Gamma_{10} + 2\Gamma_{20} + \Gamma_{21} + \Gamma_{31} + \Gamma_{32}$.

Comparing Eq. (S20b) with Eq. (S11), we can see that the amplitudes $C_1^{\pm}$ of the narrow peak are multiplied by a factor Γ_{30}/Ω compared to the amplitudes $C_0^{\pm}$ of the outer sidebands. This factor is a small quantity, thus the narrow peak can be regarded as a small correction to the outer sidebands. Whereas the linewidths of the ultranarrow peak $\lambda_1^{\pm}$ are also small quantities, so that the height of the narrow peak $C_1^{\pm}/\lambda_1^{\pm}$ can be equivalent to the height of the central peak, which makes it observable in the emission spectrum under certain conditions.

In summary, the cavity emission spectrum $S(\omega)$ can be expressed as

$$S(\omega) = \frac{|\alpha_{03}|^2}{2} \mathcal{R} \left[\frac{C_0}{\lambda_0 - i\omega} + \frac{C_0^+}{\lambda_0^+ - i\omega} + \frac{C_0^-}{\lambda_0^- - i\omega} + \frac{C_1^+}{\lambda_1^+ - i\omega} + \frac{C_1^-}{\lambda_1^- - i\omega} \right] + \mathcal{R} \left[\frac{C_2^+}{\lambda_2^+ - i\omega} + \frac{C_2^-}{\lambda_2^- - i\omega} \right], \tag{S21}$$

where $C_2^{\pm} = |\alpha_{01}|^2 \rho_{11}^{ss} + |\alpha_{13}|^2 \rho_{\pm\pm}^{ss}$. The emission spectrum consists of seven parts, where λ_0 represents the central peak linewidth of the Mollow-like triple, $\lambda_0^{\pm}$ ($\lambda_2^{\pm}$) represent the linewidths of the sidebands located at $\pm\Omega$ ($\pm\Omega/2$). Finally and most importantly, $\lambda_1^{\pm}$ (both real) represent the linewidths of the narrow peak, which are significantly smaller than any other linewidth.

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