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Spatially dependent optical bistability: supplement

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Let us consider the total electromagnetic field with the following form

$$\begin{aligned}\vec{E} = E(x,y,z) \quad &= \tfrac{1}{\sqrt{|l|!}}\left(\tfrac{\sqrt{2}\times\sqrt{x^2+y^2}}{w_{LG}}\right)^{|l|} L_p^{|l|}\left(\tfrac{2\left(x^2+y^2\right)}{w_{LG}^2}\right) \\ &\times e^{-\left(x^2+y^2\right)/w_{LG}^2} e^{il\varphi}\vec{E}_p(z)e^{-i(\omega_p t-kz)} + c.c.,\end{aligned} \tag{1}$$

and the electrical polarization as

$$\vec{P} = \frac{1}{\sqrt{|l|!}}\left(\frac{\sqrt{2}\times\sqrt{x^2+y^2}}{w_{LG}}\right)^{|l|} L_p^{|l|}\left(\frac{2\left(x^2+y^2\right)}{w_{LG}^2}\right) e^{-\left(x^2+y^2\right)/w_{LG}^2} e^{il\varphi} P(\omega_p) e^{-i(\omega_p t-kz)} + c.c., \tag{2}$$

in the general Maxwell's Eq. (10). Here, $x^2 + y^2 = r^2$ represents the radial distance from the center of beam. Under the slowly varying envelope approximation, Eq. (10) for $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ reads to

$$\begin{aligned}&B\left\{DE_p(z) + \left[2ik\tfrac{dE_p(z)}{dz} - k^2E_p(z)\right] L_p^{|l|}\left(\tfrac{2\left(x^2+y^2\right)}{w^2}\right)\right\} - \tfrac{B}{c^2}\left(-\omega_p^2\right)E_p(z)L_p^{|l|}\left(\tfrac{2\left(x^2+y^2\right)}{w^2}\right) \\ &= -\tfrac{\omega_p^2}{\varepsilon_0 c^2} BP(\omega_p)L_p^{|l|}\left(\tfrac{2\left(x^2+y^2\right)}{w^2}\right)\end{aligned}, \tag{3}$$

with

$$B = \frac{1}{\sqrt{|l|!}}\left(\frac{\sqrt{2}\times\sqrt{x^2+y^2}}{w_{LG}}\right)^{|l|} e^{-\left(x^2+y^2\right)/w_{LG}^2} e^{il\varphi} e^{-i(\omega_p t-kz)},$$

and

$$\begin{aligned}D &= \frac{1}{w^4\left(x^2+y^2\right)}\{16\left(x^2+y^2\right)^2 L_{p-2}^{|l|+2}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) \\ &+ [-8w^2\left(x^2+y^2\right) + 16\left(x^4+y^4\right) + 32x^2y^2 - 8w^2\left(x^2+y^2\right)|l|]L_{p-1}^{|l|+1}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) \\ &+ [-l^2w^4 - 4w^2\left(x^2+y^2\right) + 4\left(x^4+y^4\right) + 8x^2y^2 + w^4|l|^2 - 4w^2\left(x^2+y^2\right)|l|]L_p^{|l|}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right)\},\end{aligned}$$

By substituting the wave number $k = \omega_p/c$ into Eq. (3), the above equation reduces to

$$DE_p(z) + \left[\tfrac{2i\omega_p}{c}\tfrac{dE_p(z)}{dz}\right] L_p^{|l|}\left(\tfrac{2\left(x^2+y^2\right)}{w^2}\right) = -\tfrac{\omega_p^2}{\varepsilon_0 c^2} P(\omega_p) L_p^{|l|}\left(\tfrac{2\left(x^2+y^2\right)}{w^2}\right). \tag{4}$$

Since the bottom index of the associated Laguerre polynomial is a positive integer, we have used the associated Laguerre polynomial recurrence relation as $L_n^k(x) = L_n^{k+1}(x) - L_{n-1}^{k+1}(x)$.

The required associated Laguerre polynomials recurrence relations are

$$\begin{aligned}
L_{p-1}^{|l|+1}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) &= L_{p}^{|l|+1}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) - L_{p}^{|l|}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right), \\
L_{p-2}^{|l|+2}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) &= L_{p-1}^{|l|+2}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) - L_{p}^{|l|+1}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) + L_{p}^{|l|}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right), \\
L_{p-1}^{|l|+2}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) &= L_{p}^{|l|+2}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) - L_{p}^{|l|+1}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right).
\end{aligned} \tag{5}$$

Substituting Eq. (5) in Eq. (4), the following relation is obtained

$$FE_p(z) + \left[\tfrac{2i\omega_p}{c}\tfrac{dE_p(z)}{dz}\right] L_p^{|l|}\left(\tfrac{2\left(x^2+y^2\right)}{w^2}\right) = -\tfrac{\omega_p^2}{\varepsilon_0 c^2} P(\omega_p) L_p^{|l|}\left(\tfrac{2\left(x^2+y^2\right)}{w^2}\right), \tag{6}$$

in which

$$\begin{aligned}
F = {} & \frac{L_p^{|l|}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right)}{w^4\left(x^2+y^2\right)} \{16\left(x^2+y^2\right)^2 \left(\frac{L_p^{|l|+2}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) - 2L_p^{|l|+1}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right)}{L_p^{|l|}\left(\frac{2(x^2+y^2)}{w^2}\right)} + 1\right) \\
& + [-8w^2\left(x^2+y^2\right) + 16\left(x^4+y^4\right) + 32x^2y^2 - 8w^2\left(x^2+y^2\right)|l|] \left(\frac{L_p^{|l|+1}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right)}{L_p^{|l|}\left(\frac{2(x^2+y^2)}{w^2}\right)} - 1\right) \\
& + [-l^2w^4 - 4w^2\left(x^2+y^2\right) + 4\left(x^4+y^4\right) + 8x^2y^2 + w^4|l|^2 - 4w^2\left(x^2+y^2\right)|l|]\},
\end{aligned}$$

By simplifying Eq. (6), the dynamics response of the probe field is obtained by

$$\frac{c^2}{2i\omega_p}[GE_p(z)] + c\frac{dE_p(z)}{dz} = i\frac{\omega_p}{2\varepsilon_0}P(\omega_p), \tag{7}$$

with

$$\begin{aligned}
G = {} & \frac{1}{w^4\left(x^2+y^2\right)} \{16\left(x^2+y^2\right)^2 \left(\frac{L_p^{|l|+2}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right) - 2L_p^{|l|+1}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right)}{L_p^{|l|}\left(\frac{2(x^2+y^2)}{w^2}\right)} + 1\right) \\
& + [-8w^2\left(x^2+y^2\right) + 16\left(x^4+y^4\right) + 32x^2y^2 - 8w^2\left(x^2+y^2\right)|l|] \left(\frac{L_p^{|l|+1}\left(\frac{2\left(x^2+y^2\right)}{w^2}\right)}{L_p^{|l|}\left(\frac{2(x^2+y^2)}{w^2}\right)} - 1\right) \\
& + [-l^2w^4 - 4w^2\left(x^2+y^2\right) + 4\left(x^4+y^4\right) + 8x^2y^2 + w^4|l|^2 - 4w^2\left(x^2+y^2\right)|l|]\},
\end{aligned}$$

where $P\left(\omega_p\right)$ defines the induced polarization in the transition $|1\rangle \leftrightarrow |3\rangle$ and is given by

$$P\left(\omega_p\right) = N\mu_{31}\rho_{31}. \tag{8}$$

For a single circulation of the probe field in the cavity, it can be denoted by $E_p(0)$ and $E_p(L)$ at the entrance and exit planes of the sample, respectively (see Fig. 4(b)). For a perfectly tuned cavity, the boundary conditions in the steady state limit between the incident field E_p^I and the transmitted field E_p^T can be written as

$$E_p(L) = E_p^T/\sqrt{T}, \quad E_p(0) = \sqrt{T}E_p^I + RE_p(L). \tag{9}$$

It is deduced from Eq. (9) that the feedback mechanism due to the reflection from the mirrors is responsible for bistability. Substituting Eq. (8) into Eq. (7) and integrating Eq. (7) on the length of the atomic medium, we obtain

$$\int_{E_p(0)}^{E_p(L)} \frac{dE_p(z)}{E_p(z) + \frac{N\omega_p^2\mu_{31}\rho_{31}}{\varepsilon_0 c^2 G}} = -\frac{c}{2i\omega_p}G\int_0^L dz. \tag{10}$$

After solving the above integrals,

$$\frac{E_p(L) + \frac{N\omega_p^2\mu_{31}\rho_{31}}{\varepsilon_0 c^2 G}}{E_p(0) + \frac{N\omega_p^2\mu_{31}\rho_{31}}{\varepsilon_0 c^2 G}} = e^{-\frac{c}{2i\omega_p}GL},$$

and using the boundary conditions, Eq. (10) reads to

$$\frac{E_p^I\mu_{31}}{\hbar\sqrt{T}}e^{-\frac{c}{2i\omega_p}GL} = \left(1 - Re^{-\frac{c}{2i\omega_p}GL}\right)\frac{\mu_{31}}{\hbar T}\frac{E_p^T}{\sqrt{T}} + \frac{N\omega_p^2\mu_{31}^2\rho_{31}}{\varepsilon_0 c^2 G\hbar T}\left(1 - e^{-\frac{c}{2i\omega_p}GL}\right), \tag{11}$$

and using definitions of cooperative parameter of atoms, C, normalized input, Y, and output, X, fields, the general expression of the OB phenomenon is obtained, Eq. (11).