Supplemental Document

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Invisible non-Hermitian potentials in discrete-time photonic quantum walks: supplement

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This supplemental document contains some technical details and mathematical proofs on non-Hermitian photonic quantum walks in the fiber loop setup.

1. SCATTERING ANALYSIS: FIRST-ORDER (BORN) APPROXIMATION

In the moving reference frame (x,t), with $x = n + vt$ and $t = m$, the light dynamics in the fiber loops is described by the discrete-time coupled equations given in the main manuscript

$$f(x,t+1) = [\cos\beta f(x-v+1,t) + i\sin\beta g(x-v+1,t)]\exp[-i\varphi(x)] \tag{S1}$$
$$g(x,t+1) = i\sin\beta f(x-v-1,t) + \cos\beta g(x-v-1,t). \tag{S2}$$

A scattering wave solution to Eqs.(S1) and (S2) with quasi-energy ϵ_0 is given by $f(x,t) = F(x)\exp(-i\epsilon_0 t)$, $g(x,t) = G(x)\exp(-i\epsilon_0 t)$, where

$$\exp(-i\epsilon_0)F(x) = [\cos\beta F(x-v+1) + i\sin\beta G(x-v+1)]\exp[-i\varphi(x)] \tag{S3}$$
$$\exp(-i\epsilon_0)G(x) = i\sin\beta F(x-v-1) + \cos\beta G(x-v-1). \tag{S4}$$

We note that the quasi energy is defined apart from integer multiples than 2π, so that we take $-\pi \leq \epsilon_0 < \pi$. The potential $\varphi(x)$ is assumed to vanish as $x \to \pm\infty$ faster than $\sim 1/x$, so that the asymptotic form of $F(x)$ and $G(x)$ is given by a superposition of plane waves. For the sake of definiteness, let us assume that the incident wave is a Bloch wave belonging to the upper lattice band with wave number q_0^+, defined by the relation $\epsilon_+(q_0^+) = \epsilon_0$. Since for a drift velocity v larger than $\cos\beta$ there are not reflected waves, the asymptotic behavior of $F(x)$ and $G(x)$, as $x \to -\infty$, reads

$$\begin{pmatrix} F(x) \\ G(x) \end{pmatrix} \simeq \begin{pmatrix} \bar{F}_+(q_0^+) \\ \bar{G}_+(q_0^+) \end{pmatrix} \exp(iq_0^+ x) \tag{S5}$$

while for $x \to \infty$ one has

$$\begin{pmatrix} F(x) \\ G(x) \end{pmatrix} \simeq \sum_{\alpha,\pm} t_\alpha^\pm \begin{pmatrix} \bar{F}_\pm(q_\alpha^\pm) \\ \bar{G}_\pm(q_\alpha^\pm) \end{pmatrix} \exp(iq_\alpha^\pm x). \tag{S6}$$

In the above equations,

$$\begin{pmatrix} \bar{F}_\pm(q) \\ \bar{G}_\pm(q) \end{pmatrix} = \begin{pmatrix} i\sin\beta\exp[iq(1-v)] \\ \exp[-i\epsilon_\pm(q)] - \cos\beta\exp[iq(1-v)] \end{pmatrix} \tag{S7}$$

are the amplitudes of the Bloch eigenstates of the lattice in the upper (+) and lower (-) bands, whereas the Bloch wave numbers $q_\alpha^\pm$ are defined by the equations (see Fig.2 of the main manuscript)

$$\epsilon_+(q_\alpha^+) = \epsilon_0 + 2\pi\alpha \ , \ \ \epsilon_-(q_\alpha^-) = \epsilon_0 + 2\pi\alpha \tag{S8}$$

with $\epsilon_\pm(q) = qv \pm \mathrm{acos}(\cos\beta\cos q)$ and $\alpha = 0, \pm1, \pm2, ...$ The integer α basically represents the scattering order (channel) arising from the discrete-time nature of the dynamics. The amplitudes $t_\alpha^\pm$ are the transmission coefficients of various scattering channels, labelled by the index α, in the two bands ($\pm$). Clearly, the scattering potential is invisible provided that all amplitudes $t_\alpha^\pm$ vanish, apart from t_0^+ which should be $t_0^+ = 1$, regardless of the value of quasi energy ϵ_0, i.e. for any incident Bloch wave.

An analytical expression of the transmission amplitudes $t_\alpha^\pm$ can be obtained in the weak potential limit $|\varphi(x)| \ll 1$ using a sandard first-order (Born) approximation. In this limit, one can write $(F(x), G(x))^T \simeq (F_0(x), G_0(x))^T + (F_1(x), G_1(x))^T$, where

$$\begin{pmatrix} F_0(x) \\ G_0(x) \end{pmatrix} = \begin{pmatrix} \bar{F}_+(q_0^+) \\ \bar{G}_+(q_0^+) \end{pmatrix} \exp(iq_0^+ x) \tag{S9}$$

is the zero-order solution, corresponding to the incident wave in the absence of the scattering potential, while $(F_1(x), G_1(x))^T$ is the leading order approximation to the scattered wave, which is the solution to the coupled equations

$$\exp(-i\epsilon_0)F_1(x) - \cos\beta F_1(x - v + 1) - i\sin\beta G_1(x - v + 1) = \Theta(x) \quad \text{(S10)}$$

$$\exp(-i\epsilon_0)G_1(x) - i\sin\beta F_1(x - v - 1) - \cos\beta G_1(x - v - 1) = 0. \quad \text{(S11)}$$

where we have set

$$\Theta(x) = -i\varphi(x)[\cos\beta F_0(x - v + 1) + i\sin\beta G_0(x - v + 1)]. \quad \text{(S12)}$$

The coupled equations (S10) and (S11) should be solved with the asymptotic condition $F_1(x), G_1(x) \rightarrow 0$ as $x \rightarrow -\infty$. Within the first order approximation, the scattering potential is invisible whenever $F_1(x), G_1(x) \rightarrow 0$ as $x \rightarrow \infty$. The solution $F_1(x), G_1(x)$ can be readily found by the method of Green′s function, namely one has

$$F_1(x) = \int_{-\infty}^{\infty} d\xi\, \Theta(\xi)\mathcal{F}(x - \xi)\ ,\ \ G_1(x) = \int_{-\infty}^{\infty} d\xi\, \Theta(\xi)\mathcal{G}(x - \xi), \quad \text{(S13)}$$

where the Green′s functions $\mathcal{F}(x)$ and $\mathcal{G}(x)$ are the solution to Eqs.(S10) and (S11), with $\Theta(x)$ replaced by the delta-Dirac function $\delta(x)$ on the right hand side of Eq.(S10), and satisfying the asymptotic condition $\mathcal{F}(x), \mathcal{G}(x) \rightarrow 0$ as $x \rightarrow -\infty$. The exact form of the Green′s function can be obtained by a standard Fourier transform analysis. After some straightforward calculations one obtains

$$\mathcal{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\hat{\mathcal{F}}(q - i\delta)\exp[i(q - i\delta)x]\ ,\ \ \mathcal{G}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\hat{\mathcal{G}}(q - i\delta)\exp[i(q - i\delta)x] \quad \text{(S14)}$$

where

$$\hat{\mathcal{F}}(q - i\delta) = \frac{\exp[2iv(q - i\delta)]\,\{\exp(-i\epsilon_0) - \cos\beta\exp[-i(v + 1)(q - i\delta)]\}}{\{\exp[-i\epsilon_0 + iv(q - i\delta)] - \exp(i\theta)\}\,\{\exp[-i\epsilon_0 + iv(q - i\delta)] - \exp(-i\theta)\}} \quad \text{(S15)}$$

$$\hat{\mathcal{G}}(q - i\delta) = \frac{i\sin\beta\exp[i(v - 1)(q - i\delta)]}{\{\exp[-i\epsilon_0 + iv(q - i\delta)] - \exp(i\theta)\}\,\{\exp[-i\epsilon_0 + iv(q - i\delta)] - \exp(-i\theta)\}}. \quad \text{(S16)}$$

In the above equations, $\delta = 0^+$ is an a arbitrarily small positive number, whereas $\theta = \theta(q - i\delta)$ is defined by the relation

$$\theta(q - i\delta) = \mathrm{acos}(\cos\beta\cos(q - i\delta)). \quad \text{(S17)}$$

Note that $\hat{\mathcal{F}}(q - i\delta)$ and $\hat{\mathcal{G}}(q - i\delta)$ are analytic functions of q in the $\mathrm{Im}(q) < 0$ lower half complex plane, while they show a numerable set of poles of first order at $q = q_\alpha^\pm + i\delta$ in the $\mathrm{Im}(q) > 0$ upper half complex plane, close to the real axis. Therefore, from the Cauchy residue theorem and Eq.(S14) it readily follows that $\mathcal{F}(x)$ and $\mathcal{G}(x)$ vanish for $x \rightarrow -\infty$, and thus according to Eq.(13) one has $F_1(x), G_1(x) \rightarrow 0$ as $x \rightarrow -\infty$, as it should be. On the other hand, for $x \rightarrow \infty$ the Green′s functions do not vanish and their behavior can be calculated from Eqs.(S14-S16) using the Cauchy residue theorem, after closing the integration contour in Eq.(S14) along the upper half complex q plane. This yields

$$\mathcal{F}(x) \sim \sum_{\alpha,\pm} iR_\alpha^{\pm,F}\exp(iq_\alpha^\pm x)\ ,\ \ \mathcal{G}(x) \sim \sum_{\alpha,\pm} iR_\alpha^{\pm,G}\exp(iq_\alpha^\pm x) \quad \text{(S18)}$$

as $x \rightarrow \infty$, where $R_\alpha^{\pm,F}$ and $R_\alpha^{\pm,G}$ are the residues of $\hat{\mathcal{F}}(q - i\delta)$ and $\hat{\mathcal{G}}(q - i\delta)$ at $q = q_\alpha^\pm + i\delta$, i.e.

$$R_\alpha^{\pm,F} = \lim_{q \rightarrow q_\alpha^\pm + i\delta}(q - q_\alpha^\pm - i\delta)\hat{\mathcal{F}}(q - i\delta)\ ,\ \ R_\alpha^{\pm,G} = \lim_{q \rightarrow q_\alpha^\pm + i\delta}(q - q_\alpha^\pm - i\delta)\hat{\mathcal{G}}(q - i\delta). \quad \text{(S19)}$$

Note that the following relation holds

$$R_\alpha^{\pm,G} = -i\frac{\exp[iq_\alpha^\pm(v - 1)]\,\{\exp(-i\epsilon_0) - \cos\beta\exp[-i(v - 1)q_\alpha^\pm]\}}{\sin\beta}R_\alpha^{\pm,F} \quad \text{(S20)}$$

which readily follows from Eqs.(S15-S16) and from the expression of ϵ_0. Using Eqs.(S12), (S18) and (S20), from Eq.(S13) it then finally follows that the asymptotic behavior of $F(x) \simeq F_0(x) + F_1(x)$ and $G(x) \simeq G_0(x) + G_1(x)$ as $x \rightarrow \infty$ is given by

$$\left(\begin{array}{c} F(x) \\ G(x) \end{array}\right) \simeq \sum_{\alpha,\pm} t_\alpha^\pm \left(\begin{array}{c} \bar{F}_\pm(q_\alpha^\pm) \\ \bar{G}_\pm(q_\alpha^\pm) \end{array}\right)\exp(iq_\alpha^\pm x) \quad \text{(S21)}$$

where the transmission amplitudes $t_\alpha^\pm$ read

$$t_\alpha^\pm = \delta_{\alpha,0^+} + \exp[-i\epsilon_0 + i(q_0^+ - q_\alpha^\pm)(1-v)] \times R_\alpha^{\pm,F} \hat{\varphi}(q_\alpha^\pm - q_0^+) \tag{S22}$$

and where

$$\hat{\varphi}(q) = \int_{-\infty}^{\infty} dx \varphi(x) \exp(-iqx) \tag{S23}$$

is the Fourier transform of the scattering potential $\varphi(x)$. Since in principle the index α of scattering channels varies from $-\infty$ to ∞ and the residues $R_\alpha^{\pm,F}$ are non-vanishing, from Eq.(S22) it is clear that the scattering potential turns out to be invisible, for any incidence wave, if and only if $\hat{\varphi}(q) = 0$ for any q. This means that, even in the first-order (Born) approximation, any potential $\varphi(x)$ cannot be strictly invisible. However, in the limit of a slowly-drifting potential, i.e. for $v \rightarrow 0$, all wave numbers $q_\alpha^\pm$ diverge, expect for q_0^+ which does not depend on v. Since $|\hat{\varphi}(q)|$ vanishes as $q \rightarrow \pm\infty$, this implies that the dominant scattering channel is the one corresponding to $\alpha = 0$ on the upper ($+$) band, i.e. one has $t_\alpha^\pm \simeq 0$, with the exception of t_0^+, which reads $t_0^+ = 1 + \exp(-i\epsilon_0) R_0^{+,F} \hat{\varphi}(0)$. Hence, for a slowly-drifting potential the condition of invisibility is simply given by $\hat{\varphi}(0) = 0$, i.e.

$$\int_{-\infty}^{\infty} dx \varphi(x) = 0. \tag{S24}$$

We note that this result is valid only in first-order (Born) approximation, i.e. for a weak potential strength. As shown in the next section, for the class of Kramers-Kronig potentials the invisibility of a slowly-drifting potential holds beyond the first-order approximation, i.e. even for a strong potential. Finally, it should be mentioned that, in order to avoid reflection, the condition $v > \cos\beta$ should be satisfied. For a slowly-drifting potential ($v \rightarrow 0$) this necessarily implies $\beta \rightarrow \pi/2^-$ for the coupling angle, with $\cos\beta \simeq (\pi/2 - \beta) > v$.

2. INVISIBILITY OF SLOWLY-DRIFTING KRAMERS-KRONIG POTENTIALS

Let us consider the class of Kramers-Kronig potentials [2], i.e. such that $\varphi(x)$ is an analytic function of the complex x variable either in the upper-half complex plane $\mathrm{Im}(x) \geq 0$, or in the lower-half complex plane $\mathrm{Im}(x) \leq 0$. This is equivalent to state that the Fourier spectrum $\hat{\varphi}(q)$ is vanishing for either $q < 0$ or $q > 0$, respectively. In order to ensure that, far from the scattering region, the asymptotic states are plane (Bloch) waves, we require $\varphi(x)$ to decay at $x \rightarrow \pm\infty$ faster than $\sim 1/x$ [1]. This requirement is satisfied provided that the so-called cancellation condition $\hat{\varphi}(q=0) = 0$ holds [1], which is equivalent to Eq.(S24). For example, any complex potential of the form

$$\varphi(x) = \sum_l \frac{A_l}{(x - x_l)^{h_l}}, \tag{S25}$$

with A_l and x_l arbitrary (even stochastic) complex numbers, with the constraint $\mathrm{Im}(x_l) > 0$ for any l [or $\mathrm{Im}(x_l) < 0$ for any l], and h_l arbitrary integer numbers larger than one, is a Kramers-Kronig potential satisfying the cancellation condition (S24).

Let us consider the scattering problem of a slowly-drifting Kramers-Kronig potential, i.e the photonic quantum walk in the double limit $v \rightarrow 0$ and $\beta \rightarrow \pi/2^-$ with $v > \cos\beta$. As discussed in the previous section, in this limit the dominant scattering channel is provided by the wave number q_0^+ solely, i.e. all transmission amplitudes $t_\alpha^\pm$, with the exception of t_0^+, are negligible. For an incident plane wave coming from $x = -\infty$ of the form given by Eq.(S5), the asymptotic behavior of the solution to Eqs.(S3) and (S4) then reads

$$\begin{pmatrix} F(x) \\ G(x) \end{pmatrix} \simeq \begin{pmatrix} \bar{F}_+(q_0^+) \\ \bar{G}_+(q_0^+) \end{pmatrix} \begin{cases} \exp(iq_0^+ x) & x \rightarrow -\infty \\ t_0^+ \exp(iq_0^+ x) & x \rightarrow \infty. \end{cases} \tag{S26}$$

To find the expression of the transmission amplitude $t_0^+ = t_0^+(q_0^+)$ beyond the weak-potential (Born) approximation considered in the previous section, we can exploit the analyticity property of the potential $\varphi(x)$ in the half complex plane (either upper or lower half plane) and use the method of complex space displacement, which has been discussed and used in some previous works to demonstrate the invisibility properties of the class of Kramers-Kronig potentials (see for instance [3–5]). Let us assume, for the sake of definiteness, that $\varphi(x)$ is analytic in the upper half complex plane, i.e. for $\mathrm{Im}(x) \geq 0$. In this case the solution $F(x)$, $G(x)$ to the scattering problem [Eqs.(S3) and (S4)] can be analytically prolonged from the real x axis into such a half complex plane. In particular, let us indicate by $F(\xi, \omega) = F(x = \xi + i\omega)$ and $G(\xi, \omega) = G(x = \xi + i\omega)$ the solutions to Eqs.(S3) and (S4) on the horizontal line Γ defined by the parametric equation $x = \xi + i\omega$, with fixed $\omega \geq 0$ and $-\infty < \xi < \infty$ and with the asymptotic form defined by Eq.(S26) as $\omega \to 0^+$. The main idea of the complex spatial displacement method is to find suitable connection relations between the transmission amplitudes $t_0^+(q_0^+, \omega)$ of scattered waves on the real x axis, i.e., for $\omega = 0$, and on the line Γ, i.e. for $\omega > 0$. Interestingly, owing to the form of Eq.(S26) it readily follows that the transmission amplitude $t_0^+(q_0^+, \omega)$ does not depend on ω: this can be readily seen by formally letting $x \to \xi + i\omega$ in Eq.(S26), so that the amplitudes of both incident and transmitted waves are multiplied by the same factor $\exp(-q_0^+ \omega)$ and thus $t_0^+(q_0^+, \omega) = t_0^+(q_0^+, \omega = 0)$. Hence we can calculate $t_0^+(q_0) = t_0^+(q_0, \omega)$ by taking the limit $\omega \to \infty$. In this limit, the scattering potential on the line Γ, $\varphi(x = \xi + i\omega)$, vanishes uniformly over the entire ξ axis, and thus $\lim_{\omega \to \infty} t_0^+(q_0^+, \omega) = 1$ because in this limit on the line Γ we basically do not have any scattering potential [5]. Therefore $t_0^+(q_0^+) = 1$, which proves that the Kramers-Kronig potential $\varphi(x)$ is invisible.

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